Reordering Rule Makes OBDD Proof Systems Stronger

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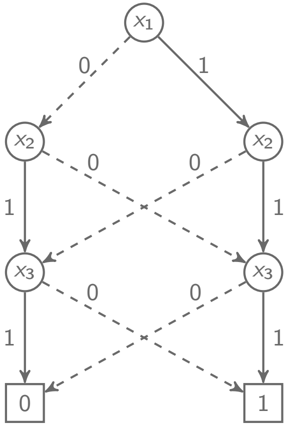
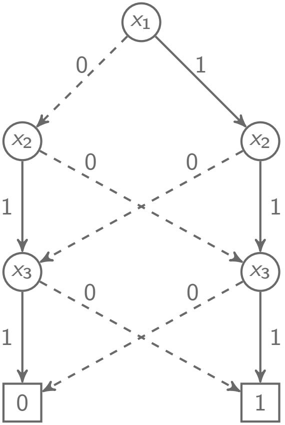
Ordered binary decision diagram (OBDD)



- OBDDs represent Boolean functions $\{0,1\}^n \to \{0,1\}$;
- π is an ordering of variables;
- if $i < j$ then $x_{\pi(j)}$ cannot appear before $x_{\pi(i)}$.



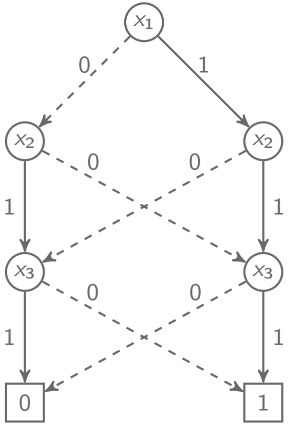
OBDD proofs of unsatisfiability:

- sequence of OBDDs: $D_1, D_2, D_3, \ldots, D_m$;
- $D_m \equiv 0$;
- OBDDs for axioms.

Rules:

- $\wedge$ (join): $D_i, D_j \Rightarrow D_k \equiv (D_i \wedge D_j)$;
- w (weakening): $D_i \Rightarrow D_j, D_i \models D_j$;
- r (reordering): $D_i \Rightarrow D_j, D_j \equiv D_i$ but orders of variables are different.

Join rule can be applied only for OBDDs in the same order.



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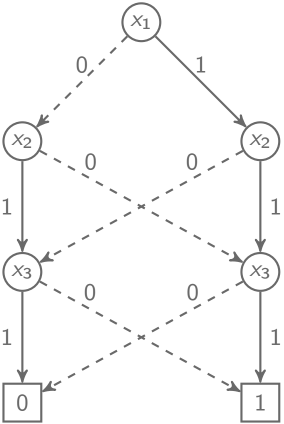

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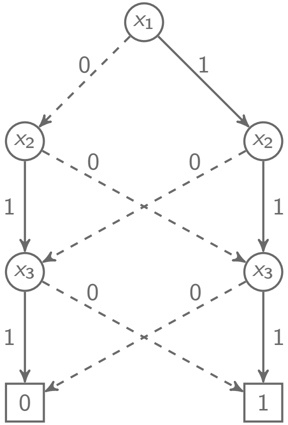

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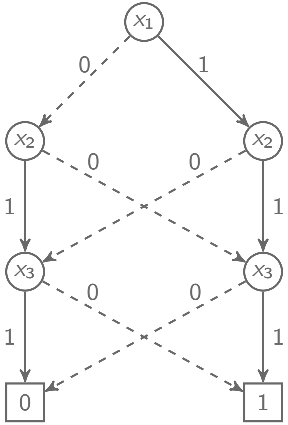

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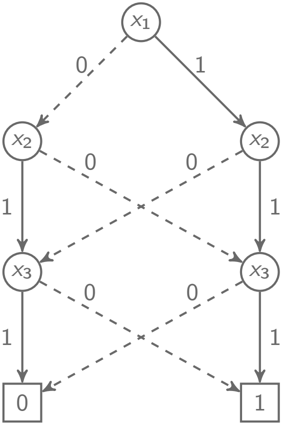

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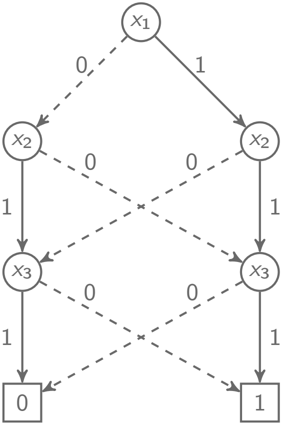

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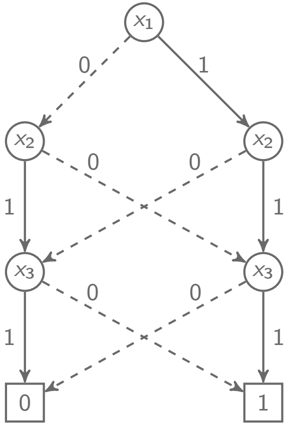

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Orders are important

- $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$;
- Alice knows $x_1, \ldots, x_n \in \{0,1\}$, Bob knows $y_1, \ldots, y_m \in \{0,1\}$;
- they want to compute $f(x,y)$;
- assume that f has an OBDD of size S in some order in that all x_i's preceed all y_j's;
- communication complexity of f is at most $\log S + 1$;
- $\mathtt{EQ} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $\mathtt{EQ}(x,y) = 1 \Leftrightarrow x = y$;
- if all x_i's preceed all y_j's in π, then size of any π-OBDD for $\mathtt{EQ}(x,y)$ is at least 2^n;
- $\exists$ short OBDD for $\mathtt{EQ}(x,y)$ in the order $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$.

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- [Atserias, Kolaitis, Vardi 04] $\mathrm{OBDD}(\wedge, \mathrm{w})$ simulates CP^*
  $\implies \mathrm{PHP}_n^{n+1}$ has proofs of poly size;
- unsatisfiable linear systems over $\mathbb{F}_2$ have short proofs;
- [Segerlind 07] $2^{n^{\Omega(1)}}$ lower bound for tree-like $\mathrm{OBDD}(\wedge, \mathrm{w})$-proofs;
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Proof of `Clique-Coloring` in semantic calculus $\mapsto$

mon. circuit, separating $(k+1)$-cliques from k-col. graphs.

Theorem (Atserias, Kolaitis, Vardi 04; Krajíček 08)

$\exists$ π *such that every* π-$\mathrm{OBDD}(\wedge, \mathrm{w})$*-proof of* `Clique-Coloring` *has size at least* $2^{n^{\delta}}$.

A hard formula for all orders? $\Phi(x) \mapsto \Psi(x, y)$.
[Krajíček 08]:

- $\forall$ orders π on x there is a substitution y_{π} such that $\Psi(x, y_{\pi})$ is isomorphic to $\Phi(x)$.
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Clique-Coloring *has a polynomial* $\mathrm{OBDD}(\wedge, \mathrm{w})$*-proof in some order.*

- Linear inequalities with small coefficients can be represented by OBDDs.
- [Hirsch, Grigoriev, Pasechnik 02] Clique-Coloring has a short LS^4 proof.
- LS^4 operates with degree 4 inequalities. The proof can be simulated by $\mathrm{OBDD}(\wedge, \mathrm{w})$ in an appropriate order.

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- $\mathrm{OBDD}(\wedge, \mathrm{w})$ *is exponentially stronger than* CP^**;*
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$\mathrm{OBDD}(\wedge, \mathrm{w}, \mathrm{r})$ is strictly stronger than $\mathrm{OBDD}(\wedge, \mathrm{w})$

- Transform $\varphi(x_1, \ldots, x_n)$ to $\tau_\varphi(z_1, \ldots, z_\ell, x_1, \ldots, x_n)$;
- $z_1, \ldots, z_\ell$ encode a permutation $\pi \in S_n$;

$$\tau(\varphi)(z_1, \ldots, z_\ell, x_1, \ldots, x_n) = \bigwedge_{\sigma \in S_n} \left[(z \text{ encodes } \sigma) \to \varphi\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \right].$$

Theorem (Segerlind 07)

$$\left. \begin{array}{l} m = \Omega(n^3) \\ \Pi \textit{ is a set of 2-ind. permut. on } [mn] \\ \exists \pi, \varphi \textit{ is hard for } \pi\text{-}\mathrm{OBDD}(\wedge, \mathrm{w}) \end{array} \right\} \Rightarrow \forall \pi, \tau(\varphi \circ \vee_m) \textit{ is hard.}$$

Theorem

$\tau_{\texttt{Clique-Coloring} \circ \vee_m}$ *is hard for* $\mathrm{OBDD}(\wedge, \mathrm{w})$ *but easy for* $\mathrm{OBDD}(\wedge, \mathrm{w}, \mathrm{r})$.

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$\tau_{\texttt{Clique-Coloring} \circ \vee_m}$ *is hard for* $\mathrm{OBDD}(\wedge, \mathrm{w})$ *but easy for* $\mathrm{OBDD}(\wedge, \mathrm{w}, \mathrm{r})$.

$\mathrm{OBDD}(\wedge, \mathrm{w}, \mathrm{r})$ is strictly stronger than $\mathrm{OBDD}(\wedge, \mathrm{w})$

- Transform $\varphi(x_1, \ldots, x_n)$ to $\tau_\varphi(z_1, \ldots, z_\ell, x_1, \ldots, x_n)$;
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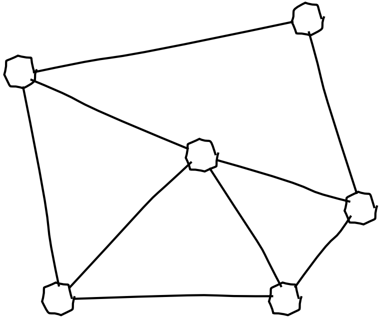

- Each node has a charge $c(v)$;
- each edge has a variable x_e;
- each node has a constraint: $\bigoplus_{e \in E_v} x_e = c(v)$;
- $\bigoplus_{v} c(v) = 1 \Rightarrow \mathrm{TS}_G$ is unsat.

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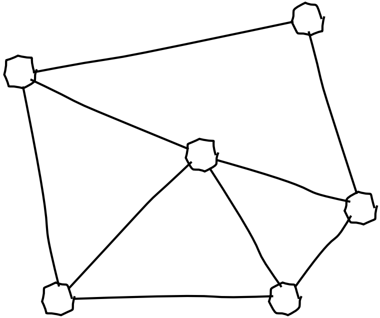
Tseitin formulas. Separation with resolution



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Lifting to CP

Theorem (Garg, Göös, Kamath, S 18)

Any CP*-proof of* $\varphi \circ \mathtt{Ind}_{n^{300}}$ *has size at least* $n^{\Theta(w(\varphi))}$*, where* $w(\varphi)$ *is a resolution width of* φ*.*

Corollary

- *Any* CP*-proof of* $\mathtt{TS}_{K_{\log(n)}} \circ \mathtt{Ind}_{n^{300}}$ *has size at least* $\log(n)^{\log^2(n)}$*;*
- *there is an* $\mathrm{OBDD}(\wedge)$*-proof of* $\mathtt{TS}_{K_{\log(n)}} \circ \mathtt{Ind}_{n^{300}}$ *of size* $\log(n)^{\log(n)}$*.*

Open problems

- Better separations between $\mathrm{OBDD}(\wedge)$ and resolution.
- Lower bounds for $\mathrm{OBDD}(\wedge, \mathrm{w}, \mathrm{r})$.
- A simulation of $\mathrm{OBDD}(\wedge, \mathrm{w})$ by Frege?