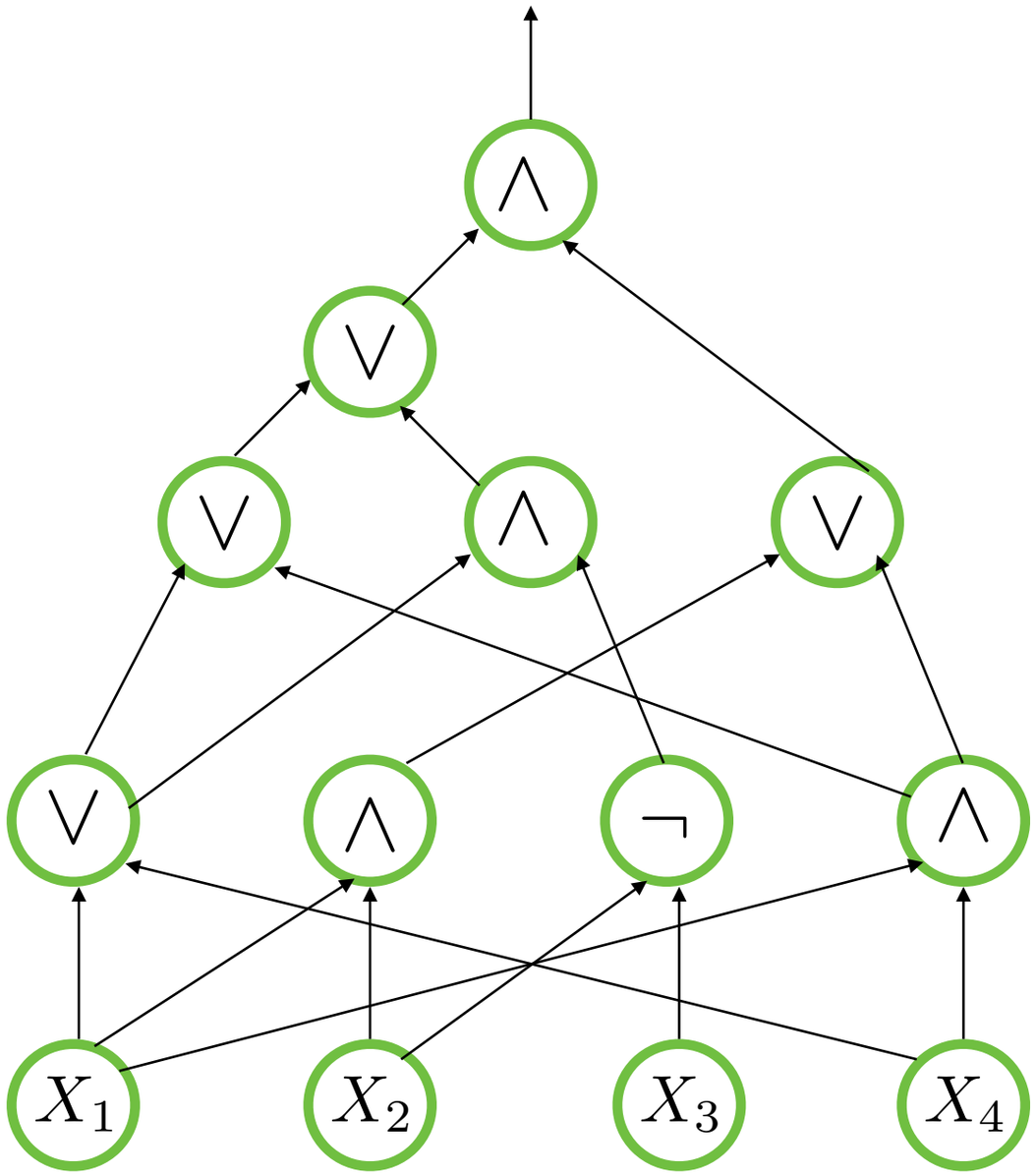
A quadratic lower bound for homogeneous algebraic branching programs

Mrinal Kumar

Speaker - Ramprasad Saptharishi

Boolean circuits



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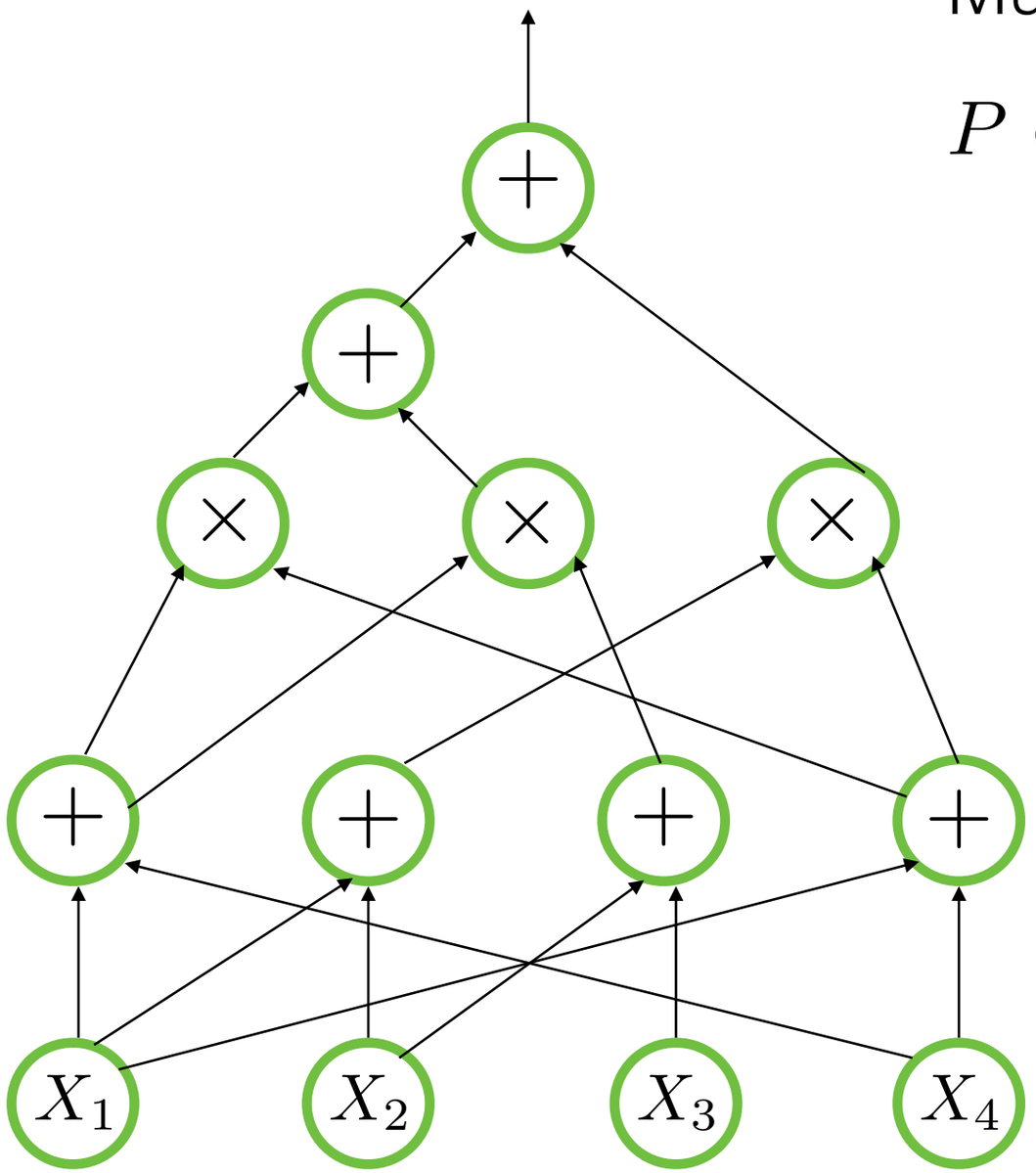
$f : \{0,1\}^n \to \{0,1\}$

Arithmetic circuits



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$$P \in \mathbb{F}[X_1, X_2, \ldots, X_n]$$

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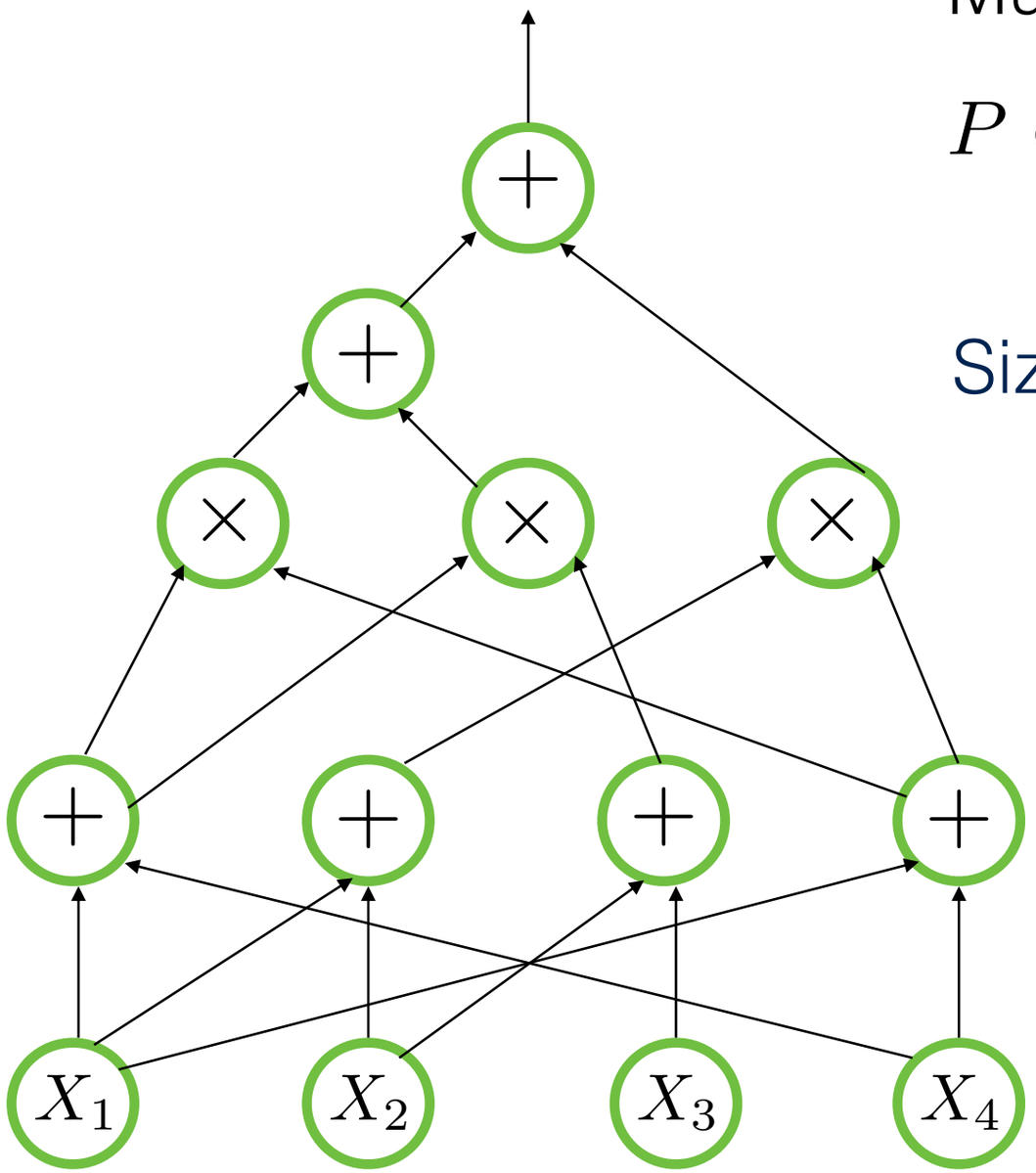

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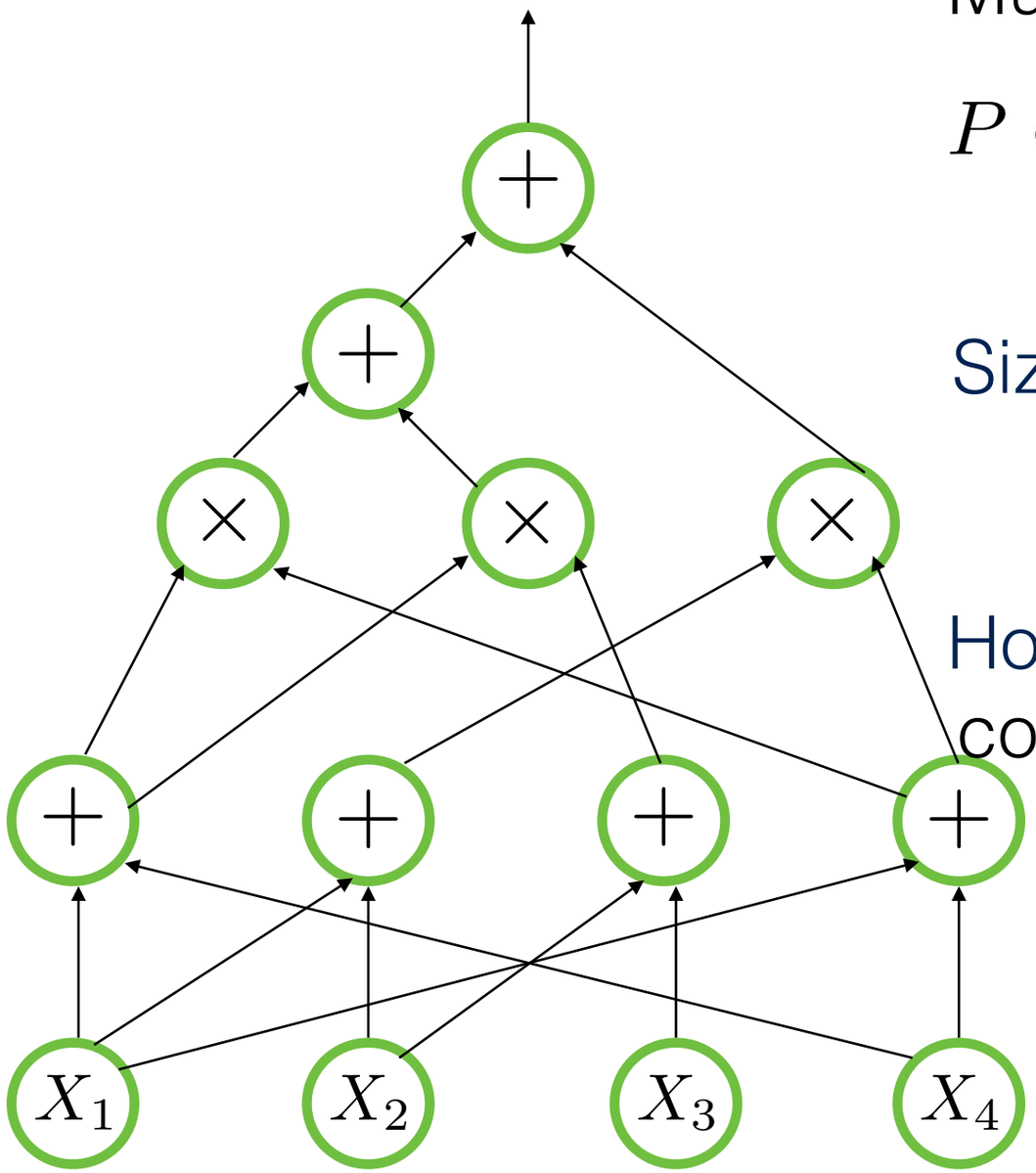


Multivariate polynomial

$P \in \mathbb{F}[X_1, X_2, \ldots, X_n]$

Size - number of gates

Homogeneous - every gate computes a homogeneous polynomial

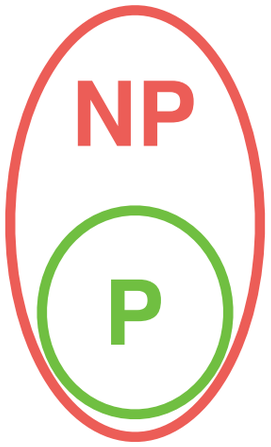
Algebraic complexity classes

- VP - Polynomials computed by poly(n) size, poly(n) degree arithmetic circuits (e.g Determinant)

- VNP - Family of explicit polynomials (e.g Permanent)

Cook's vs Valiant's hypothesis

P vs NP



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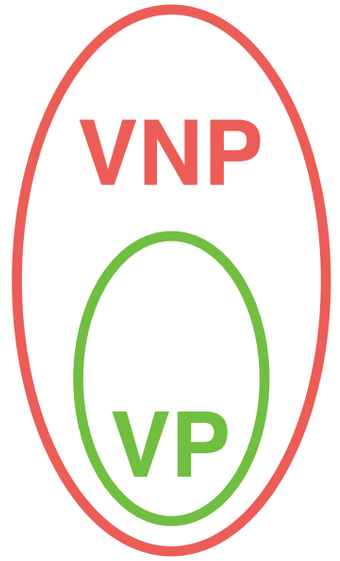

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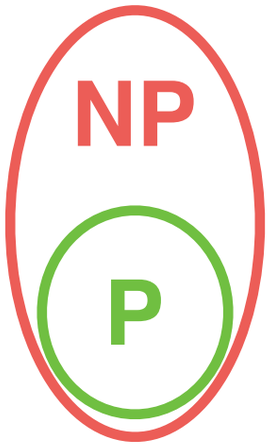

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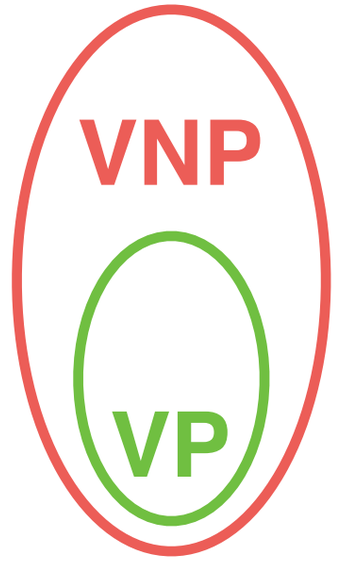

[Burgisser] Under GRH, VP = VNP implies non-uniform P = non-uniform NP.

Lower bounds for arithmetic circuits

Are there explicit polynomial families of which cannot be computed by polynomial sized arithmetic circuits ?

General lower bounds

Theorem [Strassen - 73, Baur & Strassen - 83]

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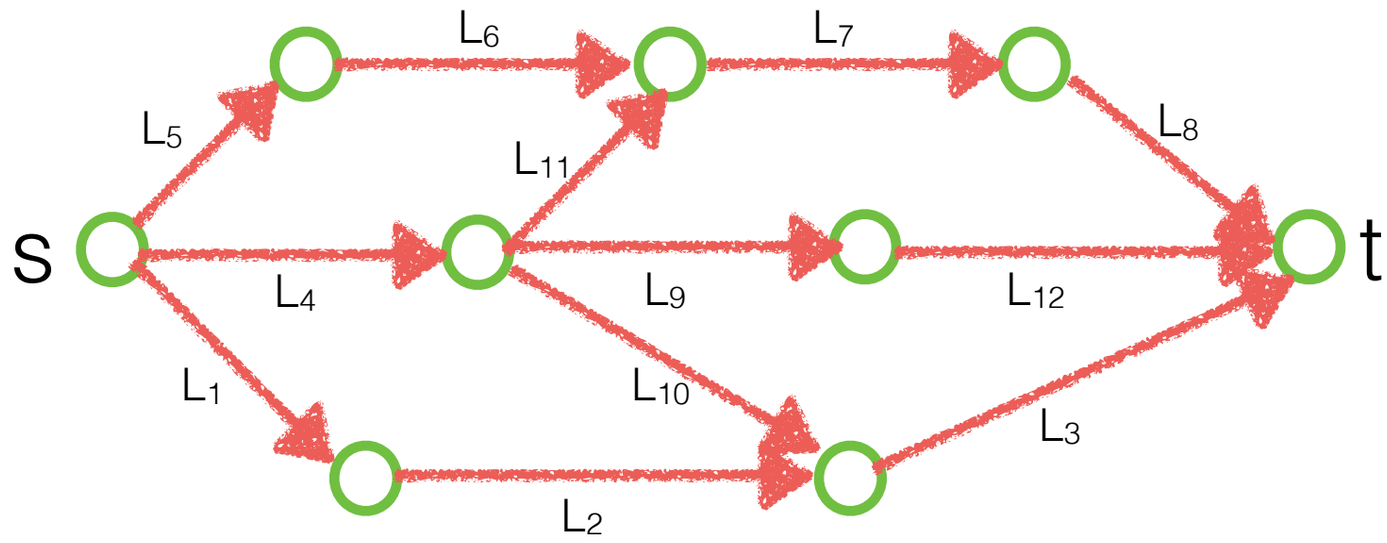
Lack of progress for this question motivates an interest in weaker models - arithmetic formula, bounded depth circuits etc.

Algebraic branching programs (ABP)

Algebraic branching programs



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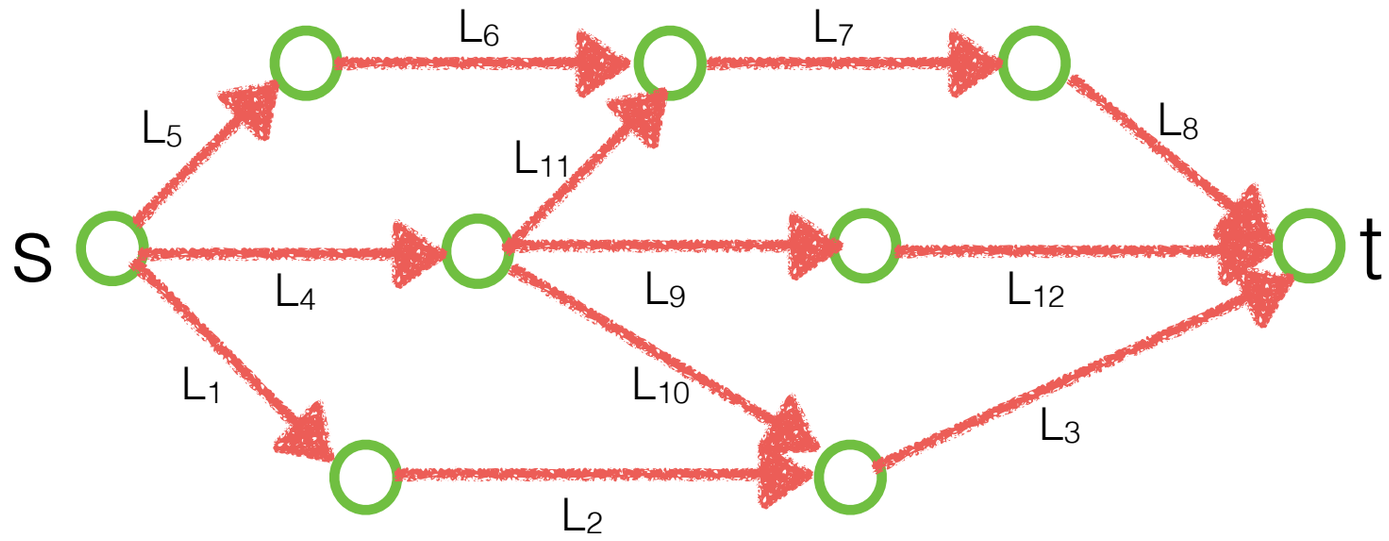

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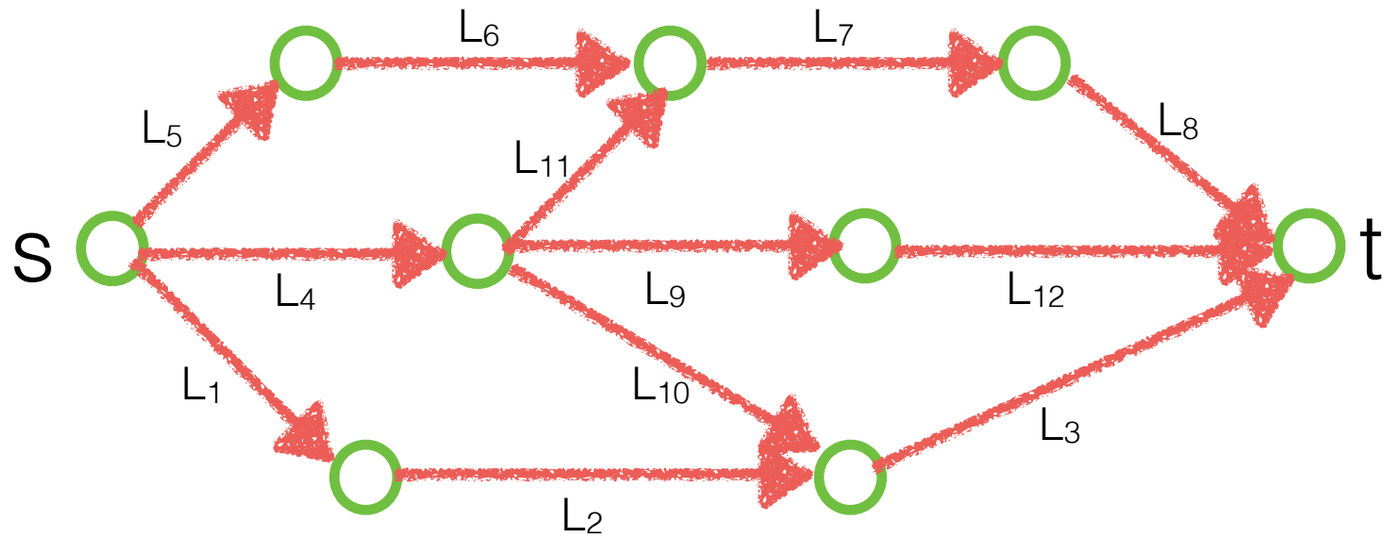

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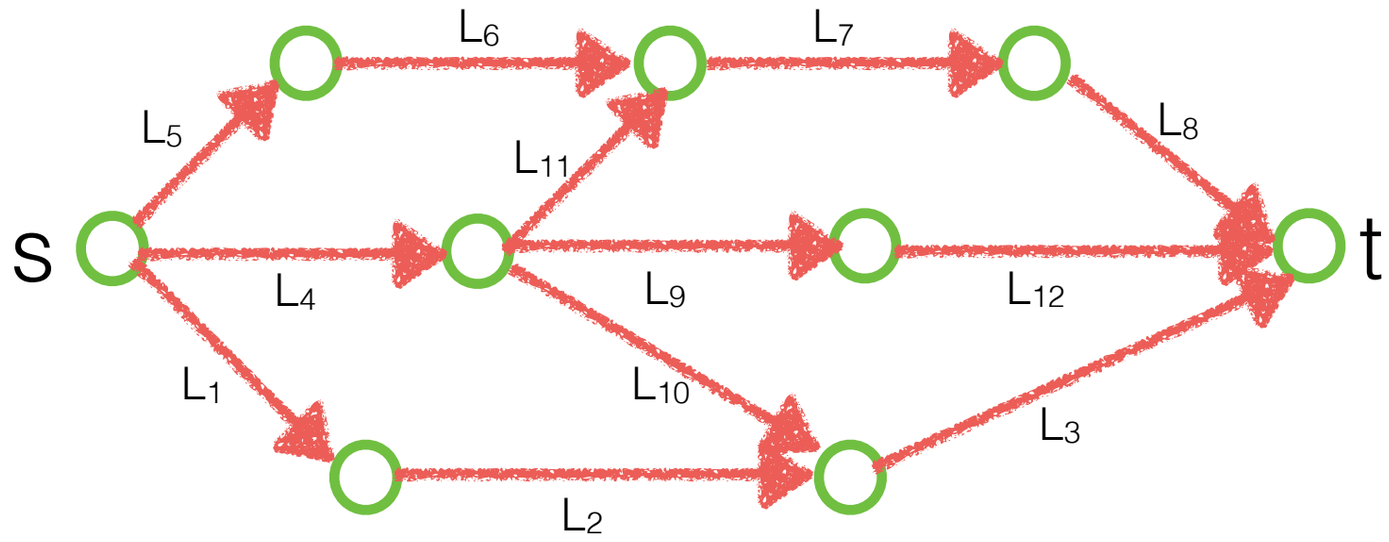

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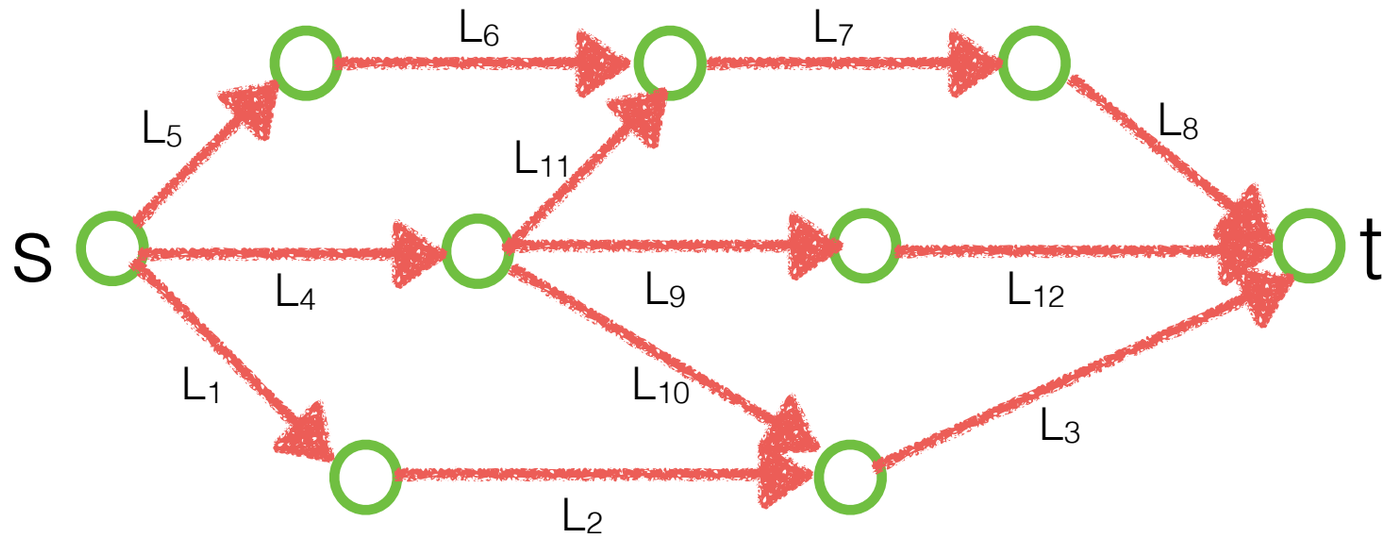

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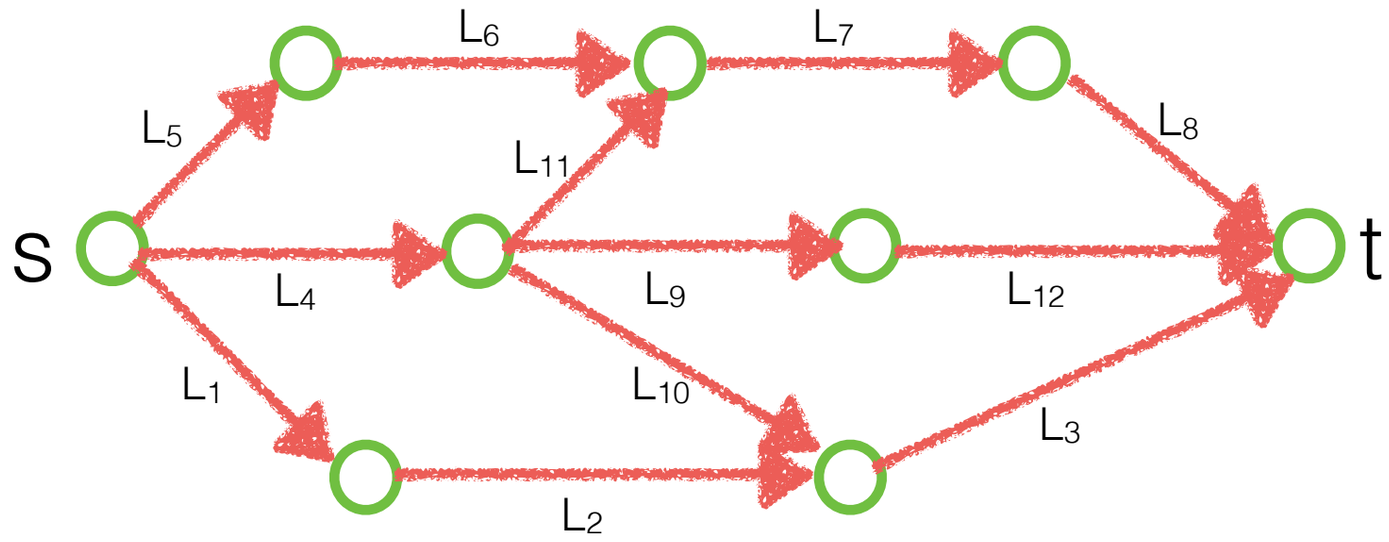

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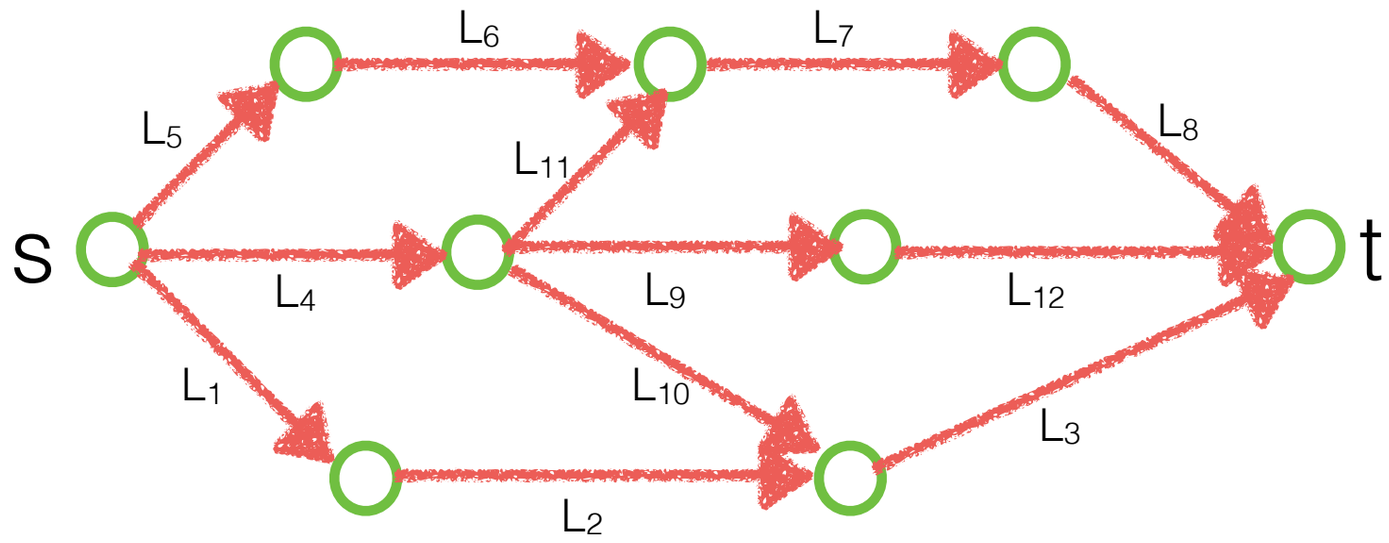

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And we do know nearly quadratic lower bounds for arithmetic formula!

Formula lower bounds

Theorem [Kalorkoti - 85]

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However, the lower bounds for ABPs continued to be just $\Omega(n \log d)$.

Results

Homogeneous ABP lower bounds

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Any homogeneous ABP which computes the polynomial $X_1^d + X_2^d + \ldots + X_n^d$ has at least $\Omega(nd)$ vertices.

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Baur-Strassen seems to imply a super linear lower bound only on the number of edges.

Also gives new proofs for….

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Our proofs are short, intuitive and essentially the same.

Proof sketches

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We work over complex numbers for rest of the talk.

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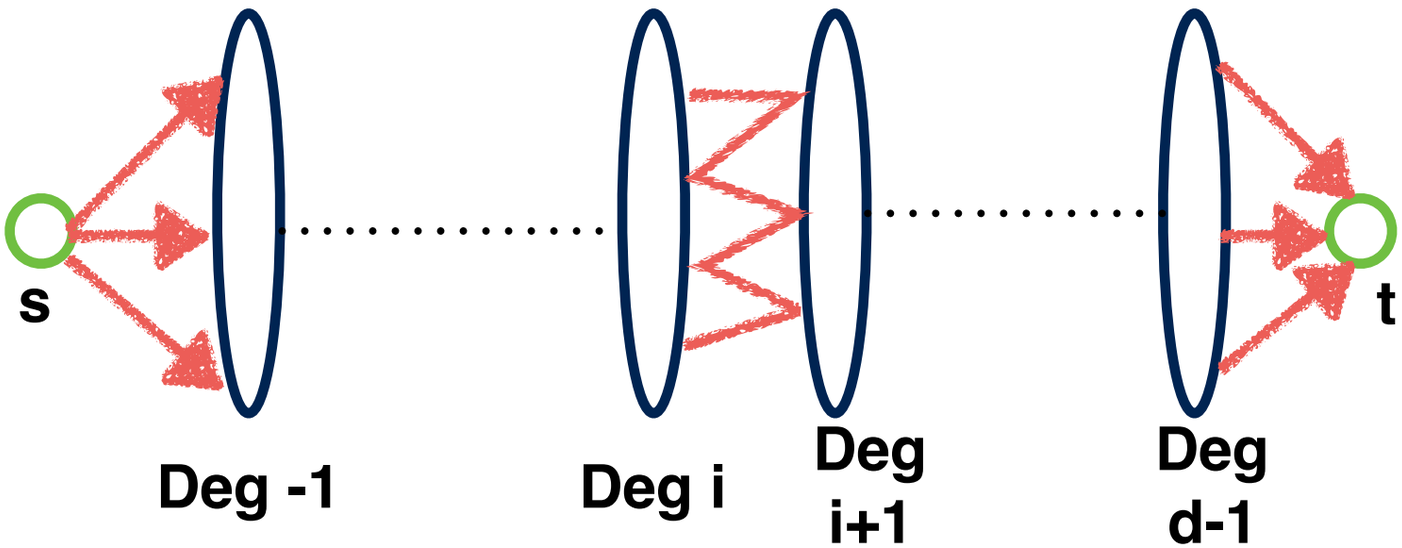

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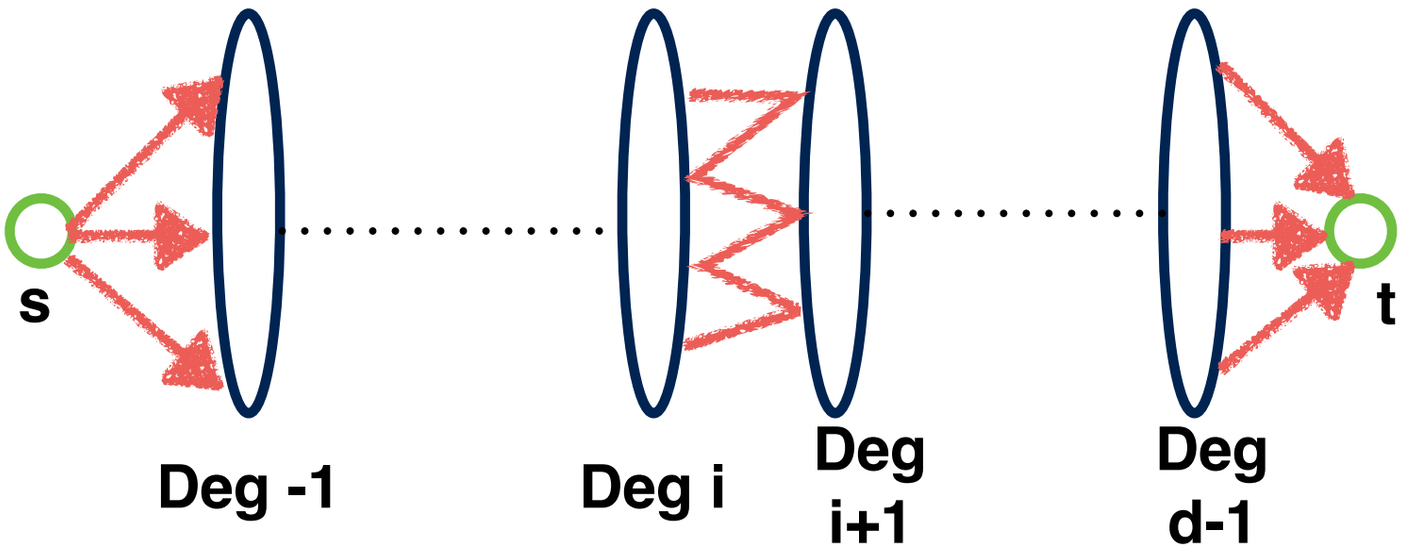

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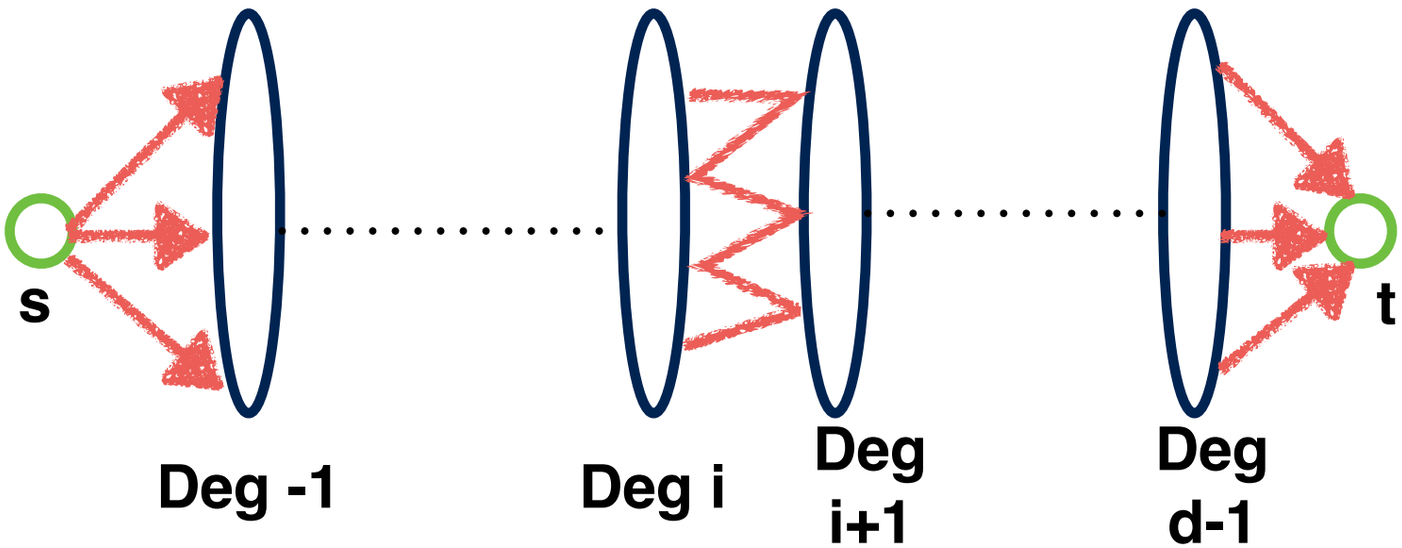

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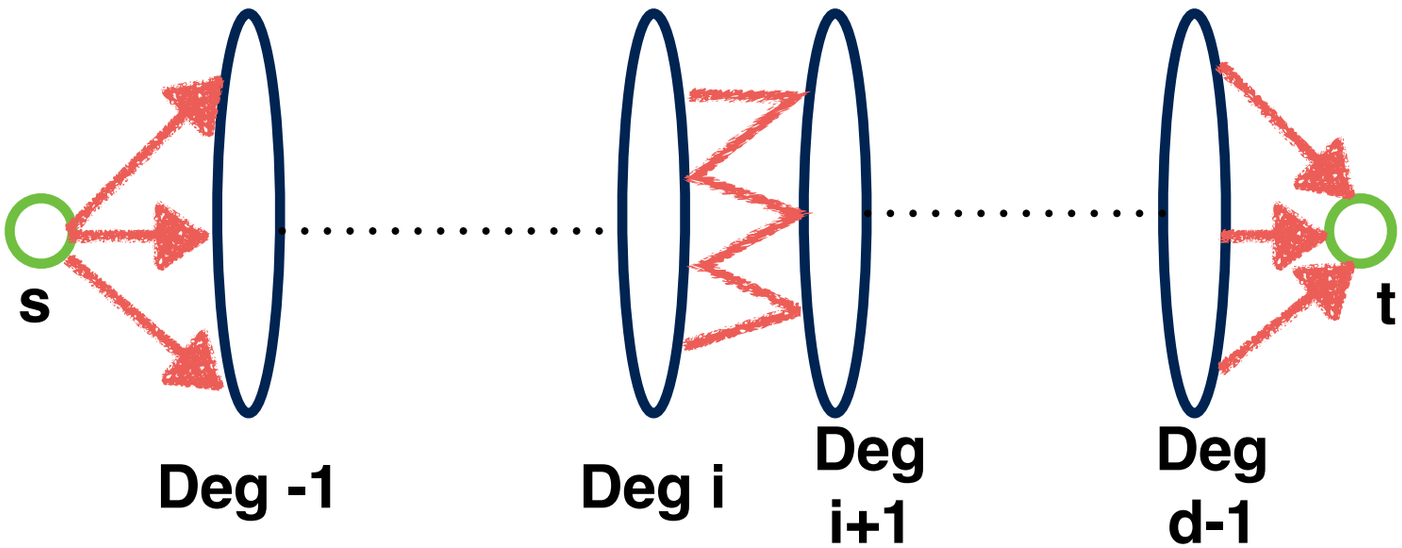

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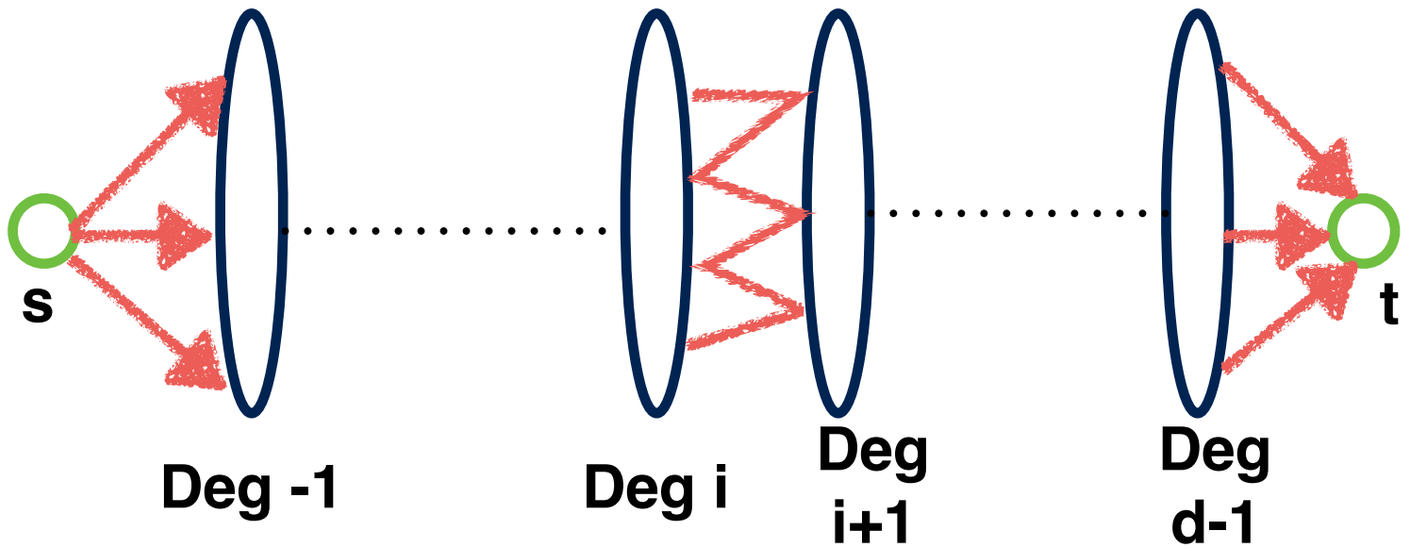

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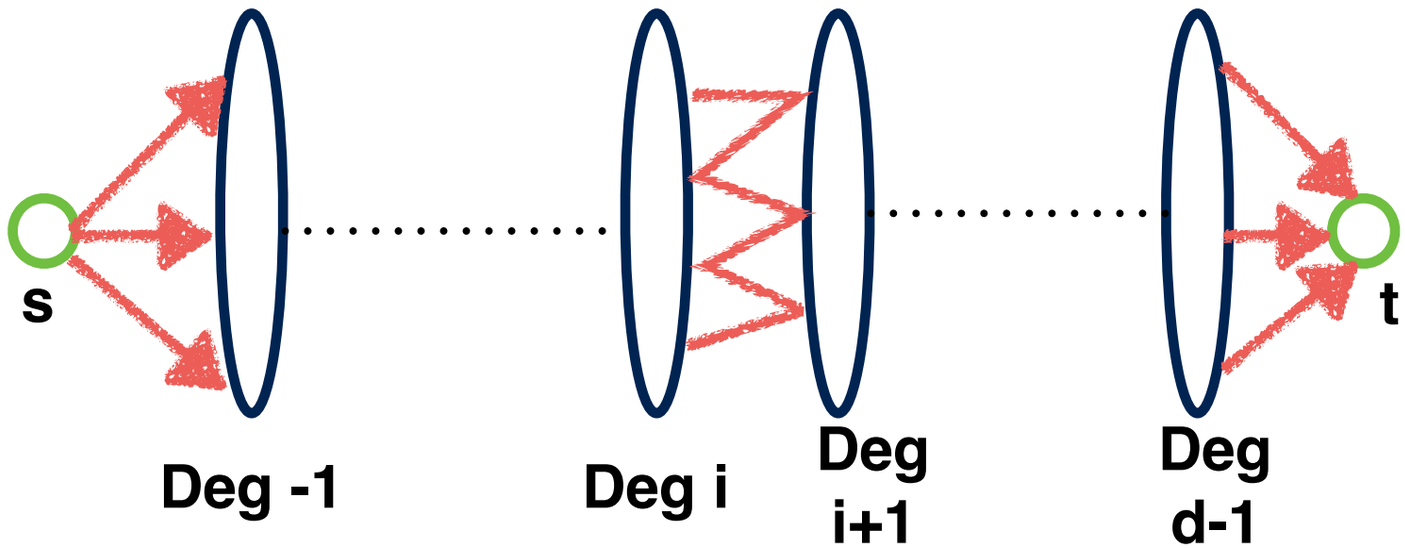

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Lemma : Each s-t vertex cut in a homogeneous ABP which computes the $X_1^d + X_2^d + \ldots + X_n^d$ must have at least $n/2$ vertices.

Proof : Let $u_1, u_2, \ldots, u_k$ be the vertices in the cut.

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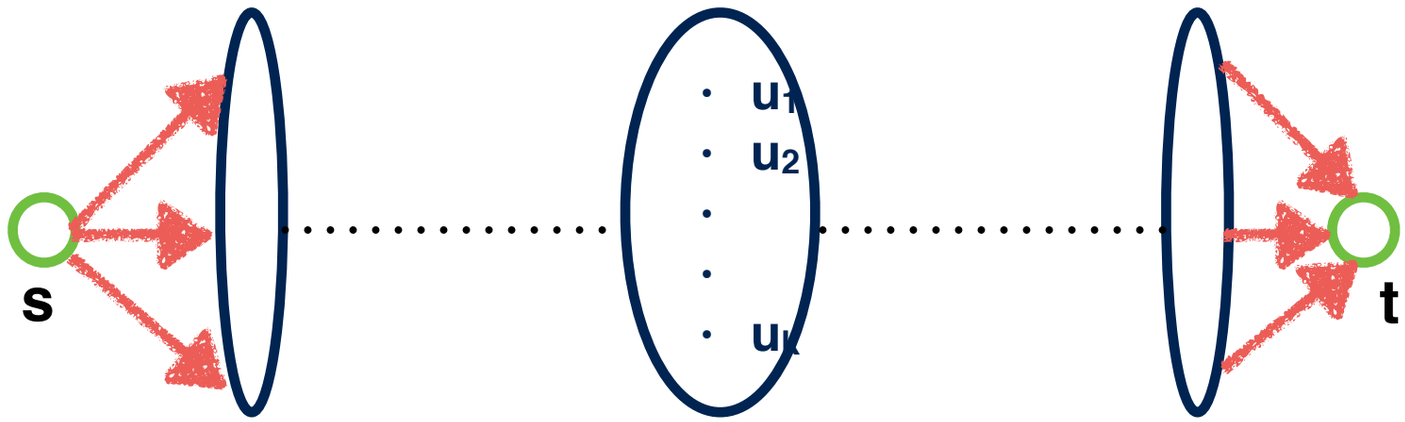

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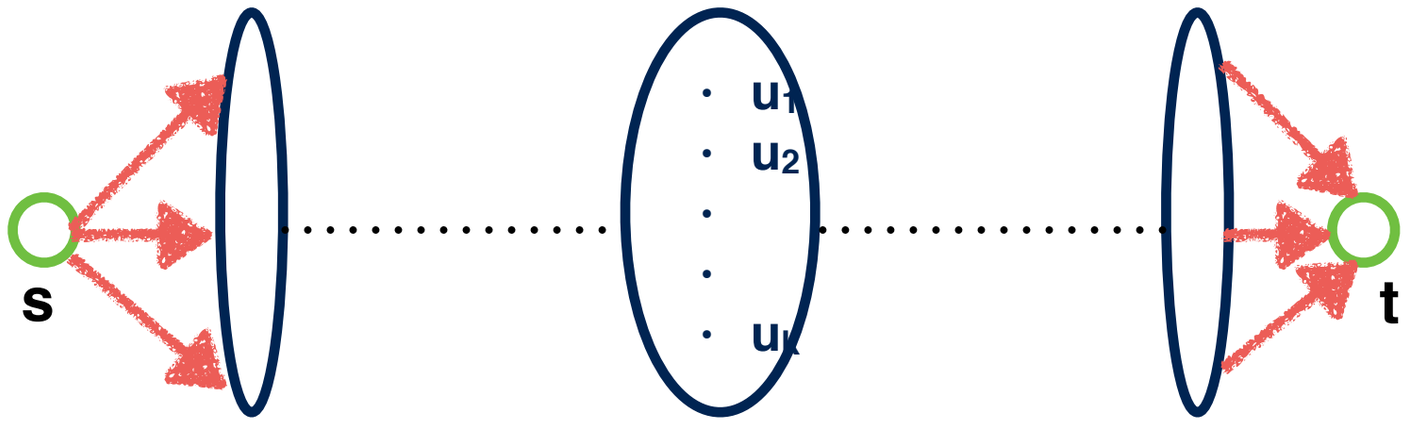

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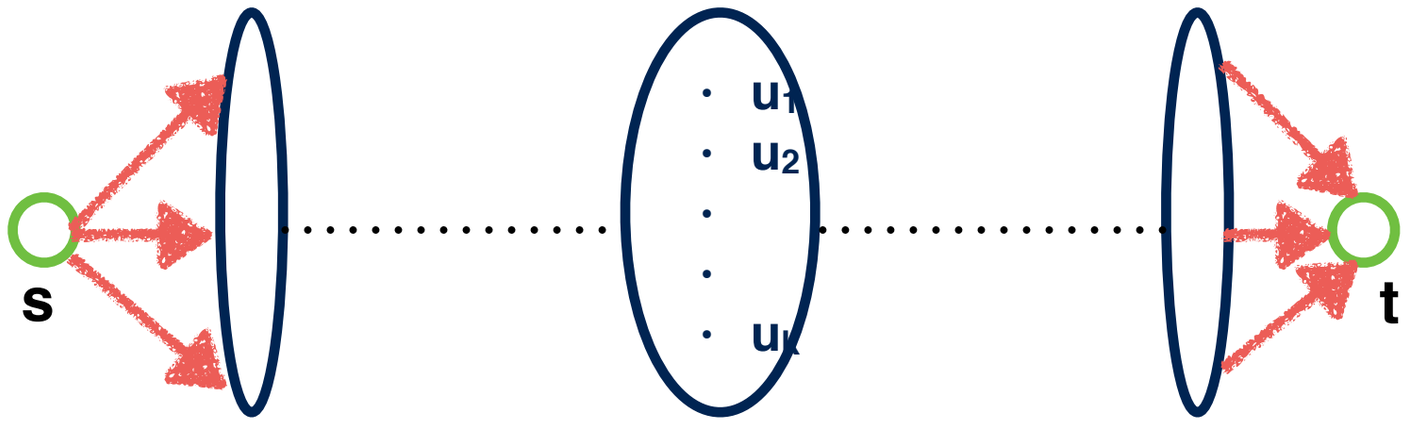

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If every such identity implied many summands, then we would be done.

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So, the dimension of zeroes of multiplicity at least two of $P_{n,d}$ is at least $n - 2k$.

But, the first order derivatives of $P_{n,d}$ are $X_1^{d-1}, X_2^{d-1}, \ldots, X_n^{d-1}$ whose only common zero is the origin. So, $n - 2k \leq 0$.

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- We have an appropriate generalization of Kayal's lemma.

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- Prove super-quadratic lower bounds for homogeneous ABPs?
    - Perhaps, we could start with trying to prove improved lower bounds for homogeneous formulas.

Thanks!