Query-to-Communication Lifting for P^{NP}

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MIT

joint work with

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Tom Watson

Riga, Latvia

CCC July 7, 2017

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

$x \to$ 

 $\leftarrow y$

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

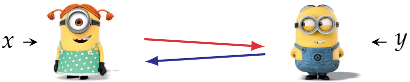
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$$

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

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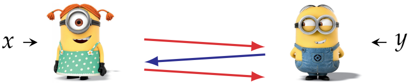

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

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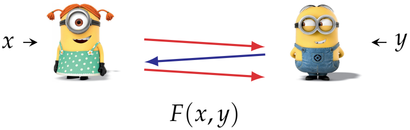
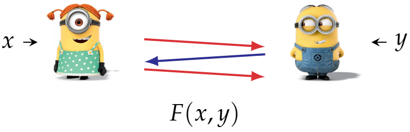


0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

(deterministic)
$\mathsf{P}^{\mathsf{cc}}(F)$

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



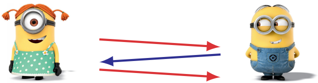
$\mathsf{P}^{\mathsf{cc}}(F) = k \implies$ partition into $\leq 2^k$ monochromatic rectangles.

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

(deterministic)
P^{cc}

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

$x \rightarrow$  $\leftarrow y$

$F(x,y)$

0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

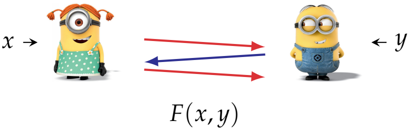
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

(deterministic)
P^{cc}

$\mathsf{BPP}^{\mathsf{cc}}$
(randomized)

randomness



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

(deterministic)
P^{cc}

↓

$\mathsf{BPP}^{\mathsf{cc}}$
(randomized)

$x \rightarrow$ 

 $\leftarrow y$

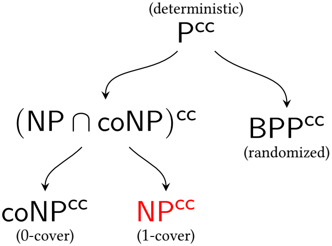
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

$x \rightarrow$ 

 $\leftarrow y$



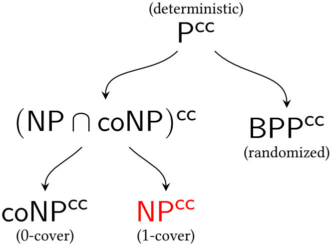
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

witness

$x \to$   $\leftarrow y$



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

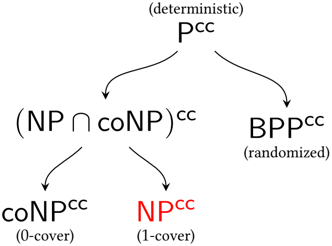
$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

witness

$x \to$   $\leftarrow y$

0 or 1   0 or 1

1 if $\begin{cases} \text{Alice says 1} \\ \text{Bob says 1} \end{cases}$



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

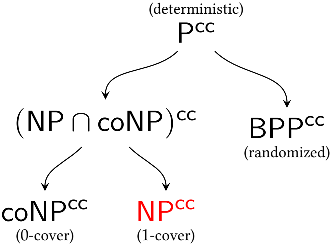
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

witness

$x \to$   $\leftarrow y$

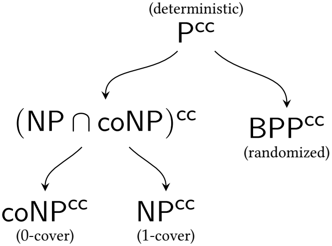
$$\mathsf{NP}^{\mathsf{cc}}(F) = k \;\Leftrightarrow\; F^{-1}(1) = \bigcup_{i \in [2^k]} R_i$$



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



$x \rightarrow$ 

 $\leftarrow y$

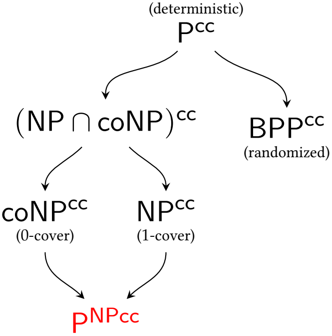
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

$x \to$ 

 $\leftarrow y$



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Communication Complexity

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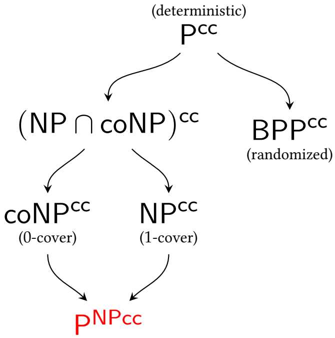
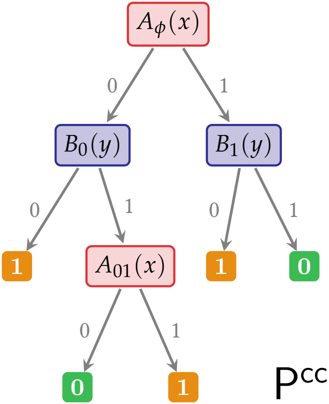


0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

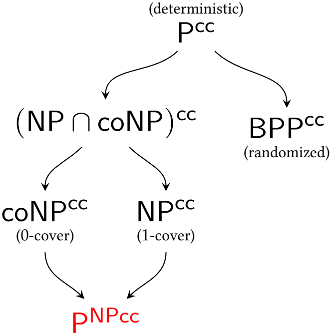
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



$\mathsf{P}^{\mathsf{cc}}(F)$

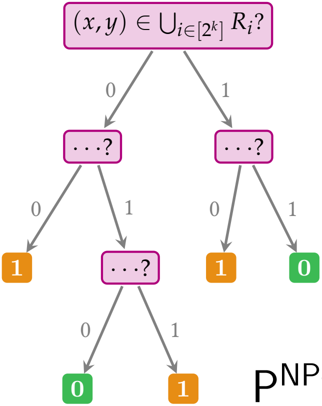
Cost : Depth of Tree



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

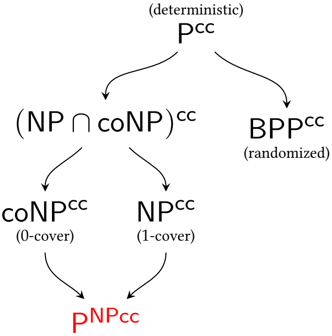
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



$$\mathsf{P}^{\mathsf{NPcc}}(F)$$

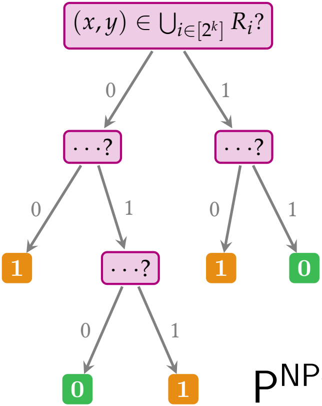
Cost : Total Oracle Cost



0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

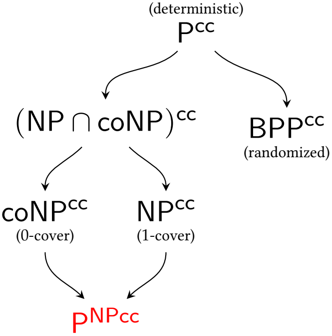
Communication Complexity

$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



$$\mathsf{P}^{\mathsf{NPcc}}(F)$$

Cost : Total Oracle Cost

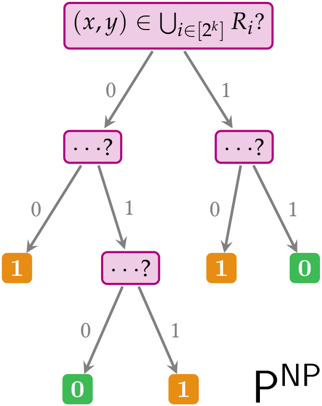


0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

??

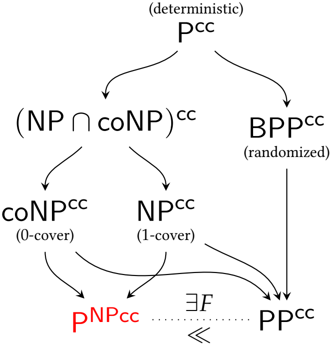
Communication Complexity

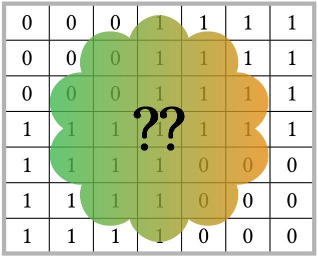
$$F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$



$$\mathsf{P}^{\mathsf{NP}\mathsf{cc}}(F)$$

Cost : Total Oracle Cost





0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	0	0	0
1	1	1	1	0	0	0
1	1	1	1	0	0	0

Monochromatic Rectangles

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\max_{\text{m.c. } R} \mu(R)}$$

0	1	0	1	1	1	1
1	0	0	0	1	0	1
0	0	1	1	0	1	1
1	1	1	1	1	0	1
1	1	1	1	0	0	0
0	1	1	1	0	1	0
1	1	0	1	0	1	0

Monochromatic Rectangles

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \ \max\limits_{\text{m.c. } R} \ \mu(R)}$$

0	1	0	1	1	1	1
1	0	0	0	1	0	1
0	0	1	1	0	1	1
1	1	1	1	1	0	1
1	1	1	1	0	0	0
0	1	1	1	0	1	0
1	1	0	1	0	1	0

Monochromatic Rectangles

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Questions about $\mathsf{mon}(F)$

- Log-Rank Conjecture [NW94]
  $\forall F \ : \ \mathsf{mon}(F) \leq \log^{O(1)} \mathsf{rk}(F)$

Monochromatic Rectangles

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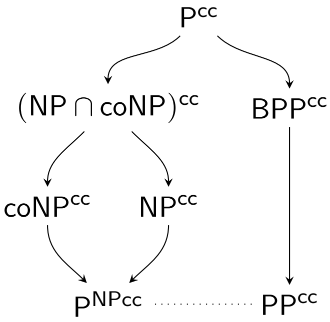

Monochromatic Rectangles

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \; \max\limits_{\text{m.c. } R} \; \mu(R)}$$

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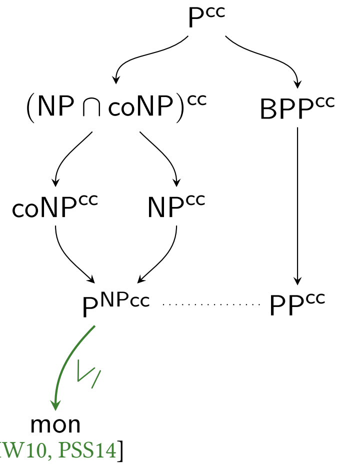

Monochromatic Rectangles

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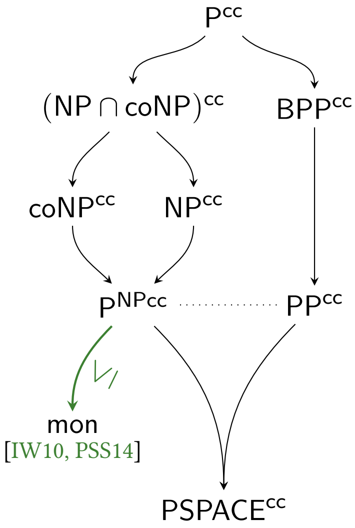
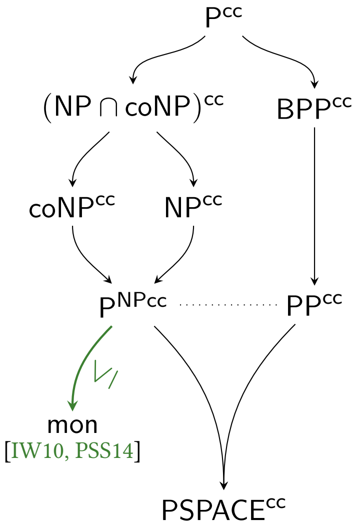


Monochromatic Rectangles

$$\mathsf{mon}(F) \overset{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \ \max\limits_{\text{m.c. } R} \ \mu(R)}$$

Questions about $\mathsf{mon}(F)$

- Log-Rank Conjecture [NW94]
  $\forall F \ : \ \mathsf{mon}(F) \leq \log^{O(1)} \mathsf{rk}(F)$
- Protocols from mon?
  $\forall F : \mathsf{PSPACE}^{\mathsf{cc}}(F) \leq \mathsf{mon}(F)^{O(1)}$



Monochromatic Rectangles

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min_{\text{product } \mu} \max_{\text{m.c. } R} \mu(R)}$$

Questions about $\mathsf{mon}(F)$

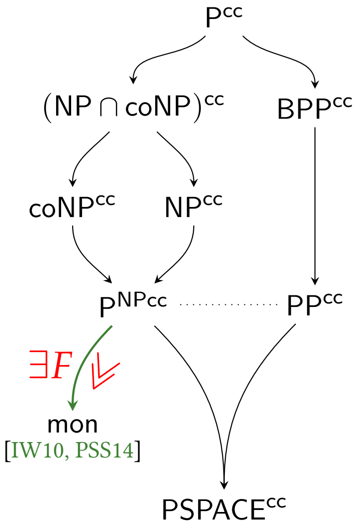
- Log-Rank Conjecture [NW94]
  $\forall F \ : \ \mathsf{mon}(F) \leq \log^{O(1)} \mathsf{rk}(F)$
- Protocols from mon?
  $\forall F : \mathsf{PSPACE}^{\mathsf{cc}}(F) \leq \mathsf{mon}(F)^{O(1)}$

This work!

$$\exists F \ : \ \mathsf{mon}(F) \leq (\log n)^{O(1)}$$
$$\mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

i.e. refutes:

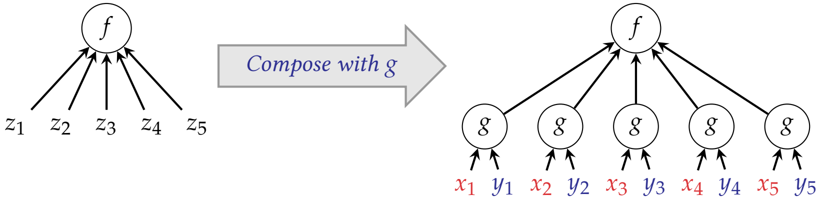
$$\forall F \ : \ \mathsf{P}^{\mathsf{NPcc}}(F) \leq \mathsf{mon}(F)^{O(1)}$$



Composed Functions

$f : \{0,1\}^n \to \{0,1\}$

$f \circ g^n : X^n \times Y^n \to \{0,1\}$



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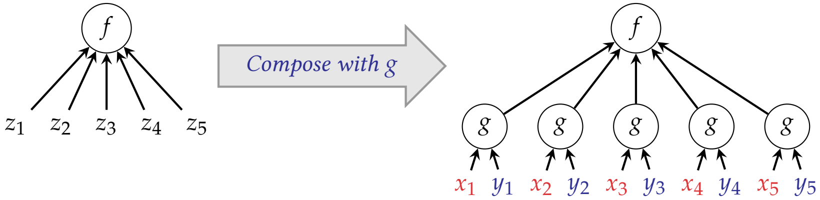


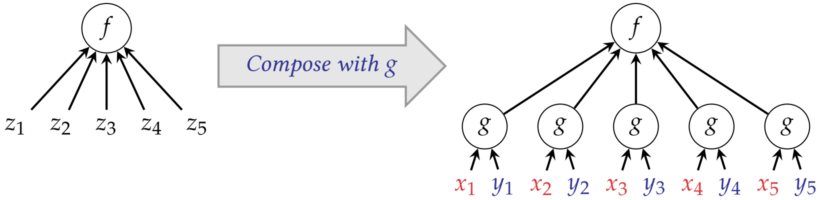
Gadget

$g : X \times Y \to \{0,1\}$ is a *small* gadget.

- Alice gets $x \in X$
- Bob gets $y \in Y$

Composed Functions

$f : \{0,1\}^n \to \{0,1\}$ $\qquad$ $f \circ g^n : X^n \times Y^n \to \{0,1\}$



EXAMPLES

- Set Disjointness : $(\neg\mathsf{OR}) \circ \mathsf{AND}^n$
- Inner Product : $\mathsf{XOR} \circ \mathsf{AND}^n$
- Equality : $\mathsf{AND} \circ (\neg\mathsf{XOR})^n$

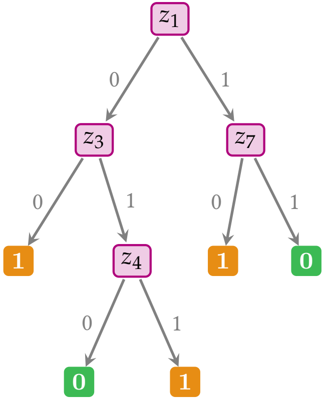
GADGET

$g : X \times Y \to \{0,1\}$ is a *small* gadget.

- Alice gets $x \in X$
- Bob gets $y \in Y$

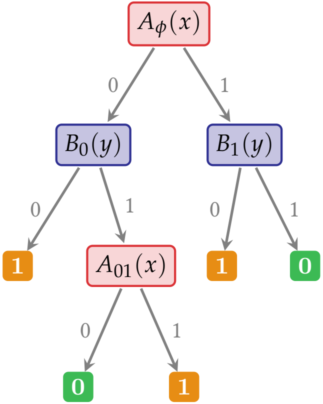
Query vs. Communication Complexity

$f : \{0,1\}^n \to \{0,1\}$



Depth of Decision Tree

$F : X^n \times Y^n \to \{0,1\}$



Depth of Protocol Tree

P

Query vs. Communication Complexity

$f : \{0,1\}^n \to \{0,1\}$

$F : X^n \times Y^n \to \{0,1\}$

$$\begin{aligned} & z_1\overline{z_3}z_5 \\ \vee\ & \overline{z_2}z_4\overline{z_7} \\ \vee\ & z_1z_2\overline{z_5} \\ \vee\ & \cdots \\ \vee\ & \cdots \end{aligned}$$

$$(x, y) \in \bigcup_{i \in [2^k]} R_i \ ?$$

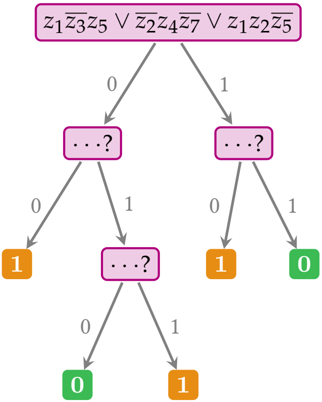
NP

Width of DNF

log #Rectangles

Query vs. Communication Complexity

$f : \{0,1\}^n \to \{0,1\}$



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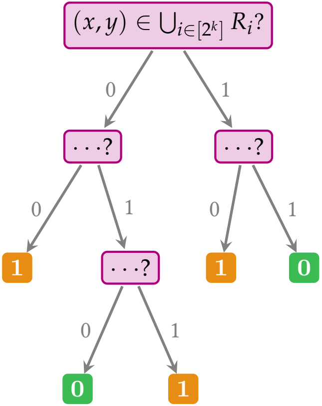


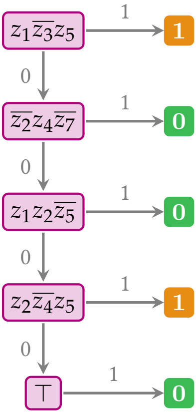
P^{NP}

Sum of DNF widths

Sum of log #Rectangles

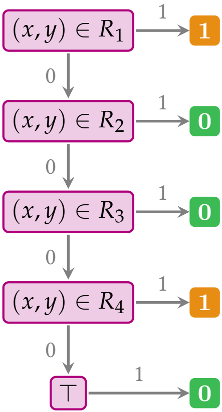
Query vs. Communication Complexity

$f : \{0,1\}^n \to \{0,1\}$



Width of DNF [Riv '87]

$F : X^n \times Y^n \to \{0,1\}$



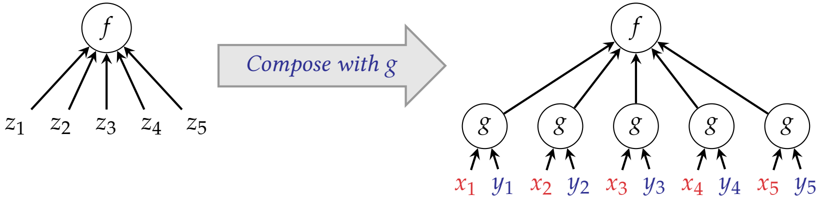
log #Rectangles [PSS '14]

DL

(Decision Lists)
quadratically
equivalent
to P^{NP}

Lifting Theorem!

$f : \{0,1\}^n \to \{0,1\}$ $\qquad$ $f \circ g^n : X^n \times Y^n \to \{0,1\}$



Lifting Theorem!

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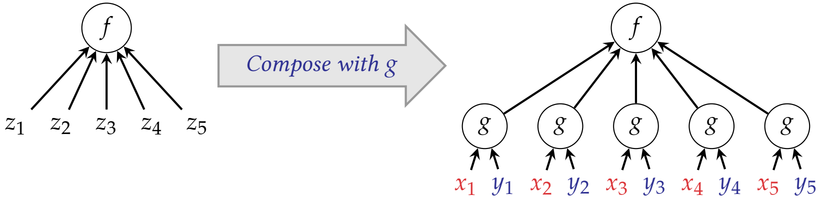
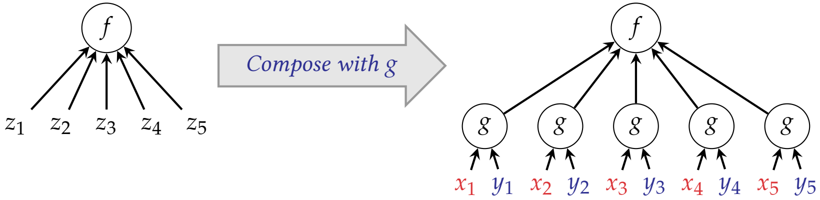


$$\mathsf{P}^{\mathsf{cc}}(f \circ g^n) \;\leq\; O(\mathsf{P}^{\mathsf{dt}}(f) \cdot b)$$

Lifting Theorem!

$f : \{0,1\}^n \to \{0,1\}$

$f \circ g^n : X^n \times Y^n \to \{0,1\}$



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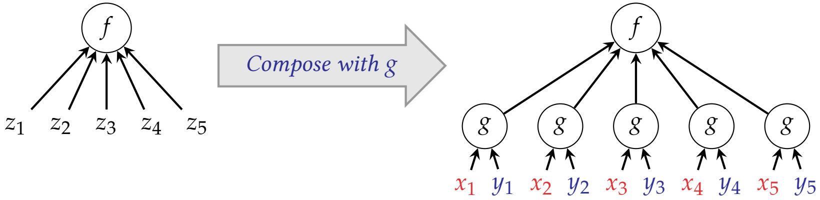


Lifting Theorem Template

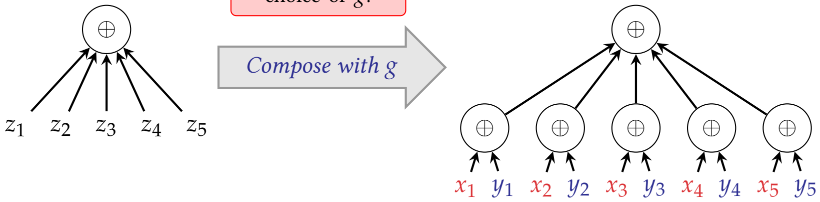
$$\mathsf{M}^{\mathsf{cc}}(f \circ g^n) \;\geq\; \Omega(\mathsf{M}^{\mathsf{dt}}(f) \cdot b)$$

Lifting Theorem!

$f : \{0,1\}^n \to \{0,1\}$

Need careful choice of g!

$f \circ g^n : X^n \times Y^n \to \{0,1\}$



Lifting Theorem Template

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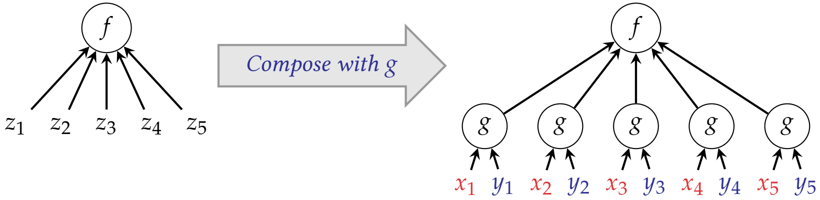

Lifting Theorem Template

Fixed g, such that for all f:

$$\mathsf{M}^{\mathsf{cc}}(f \circ g^n) \;\geq\; \Omega(\mathsf{M}^{\mathsf{dt}}(f) \cdot b)$$

Lifting Theorem!

M	Query model	Communication model	References
P	deterministic	deterministic	[RM99, GPW15, dRNV16 HHL16, WYY17, CKLM17]
NP	nondeterministic	nondeterministic	[GLM+15, Göö15]
many	polynomial degree	rank	[SZ09, She11, RS10, RPRC16]
many	conical junta degree	nonnegative rank	[GLM+15, KMR17]
	Sherali–Adams	LP extension complexity	[CLRS16, KMR17]
	sum-of-squares	SDP extension complexity	[LRS16]

Lifting Theorem Template

Fixed g, such that for all f:

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Indexing Gadget

$$g : [m] \times \{0,1\}^m \to \{0,1\}$$

$$g(x,y) = y_x$$

Lifting Theorem Template

Fixed g, such that for all f:

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Lifting Theorem!

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Indexing Gadget

$$g : [m] \times \{0,1\}^m \to \{0,1\}$$

$$g(x,y) = y_x$$

$m = n^4$

Lifting Theorem for DL & P^{NP}

- $\mathsf{DL}^{\mathsf{cc}}(f \circ g^n) \geq \mathsf{DL}^{\mathsf{dt}}(f) \cdot \Omega(\log n)$
- $\mathsf{P}^{\mathsf{NPcc}}(f \circ g^n) \geq \sqrt{\mathsf{P}^{\mathsf{NPdt}}(f) \cdot \Omega(\log n)}$

Outline of Main Application

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \max\limits_{\text{m.c. } R} \mu(R)}$$

$$\exists\, F$$

$$\mathsf{mon}(F) \leq (\log n)^{O(1)} \qquad \mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

Outline of Main Application

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \max\limits_{\text{m.c. } R} \mu(R)}$$

$$\exists\, F$$

$$\mathsf{mon}(F) \leq (\log n)^{O(1)}$$

$$\mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

$$\Updownarrow \quad \text{[PSS'14]}$$

$$\mathsf{DL}^{\mathsf{cc}}(F) \geq n^{\Omega(1)}$$

Outline of Main Application

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \max\limits_{\text{m.c. } R} \mu(R)}$$

$$\exists\, F = f \circ \mathsf{Ind}_m^n$$

$$\mathsf{mon}(F) \leq (\log n)^{O(1)}$$

$$\mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

$$\Updownarrow \quad \text{[PSS'14]}$$

$$\mathsf{DL}^{\mathsf{cc}}(F) \geq n^{\Omega(1)}$$

$$\Updownarrow \quad \text{[Lifting Theorem]}$$

$$\mathsf{DL}^{\mathsf{dt}}(f) \geq n^{\Omega(1)}$$

Outline of Main Application

$$\mathsf{mon}(F) \overset{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \max\limits_{\text{m.c. } R} \mu(R)}$$

$$\exists\, F = f \circ \mathsf{Ind}_m^n$$

1

$$\mathsf{mon}(F) \leq (\log n)^{O(1)}$$

3

$$\mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

$\Updownarrow$ [PSS'14]

$$\mathsf{DL}^{\mathsf{cc}}(F) \geq n^{\Omega(1)}$$

$\Updownarrow$ [Lifting Theorem]

$$\mathsf{DL}^{\mathsf{dt}}(f) \geq n^{\Omega(1)}$$

2

Hard Query Function f

	1					
			1			
				1		
		1				
					1	
1						
1						

$\forall \cdot$ US-complete f

$$z \in \{0,1\}^{\sqrt{n}\times\sqrt{n}}$$

$$f(z) \stackrel{\text{def}}{=} \mathbb{1}\left\{\begin{matrix}\text{Every row has} \\ \text{a unique "1"}\end{matrix}\right\}$$

$\mathsf{DL}^{\mathsf{dt}}(f) \geq \sqrt{n}$

$$f(z) \stackrel{\text{def}}{=} \mathbb{1}\left\{\begin{array}{c}\text{Every row has}\\ \text{a unique “1”}\end{array}\right\}$$

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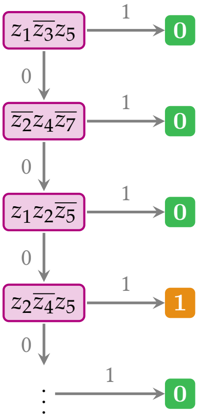

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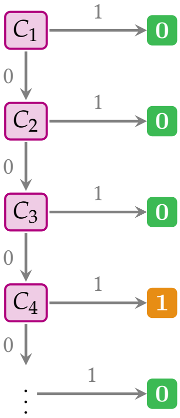

$\mathsf{width}(C_i) \leq k$

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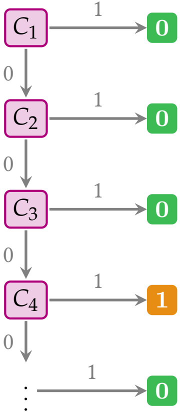

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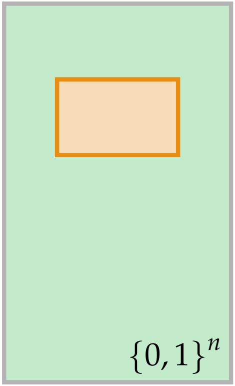

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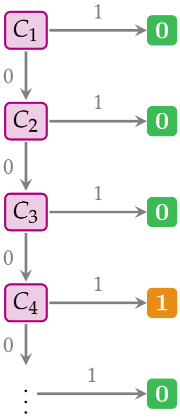

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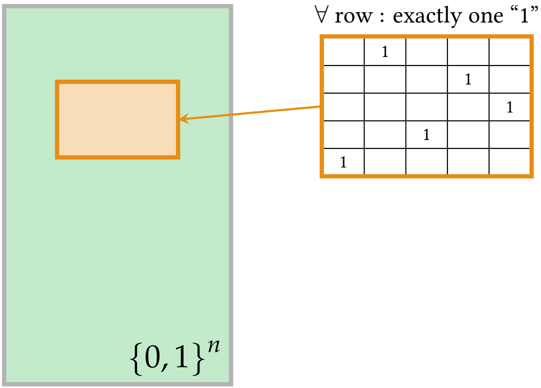

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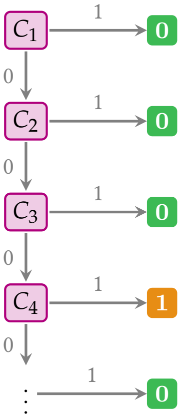

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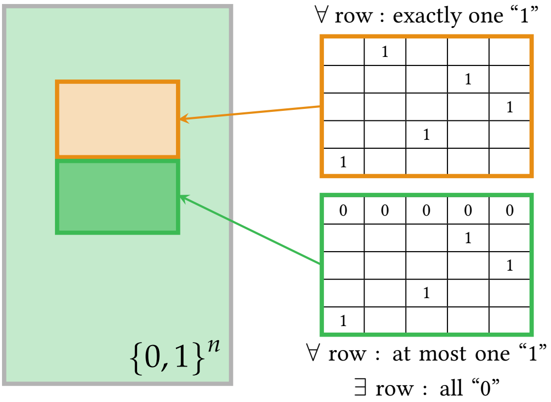

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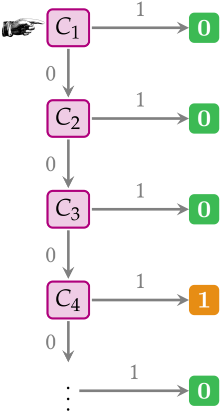

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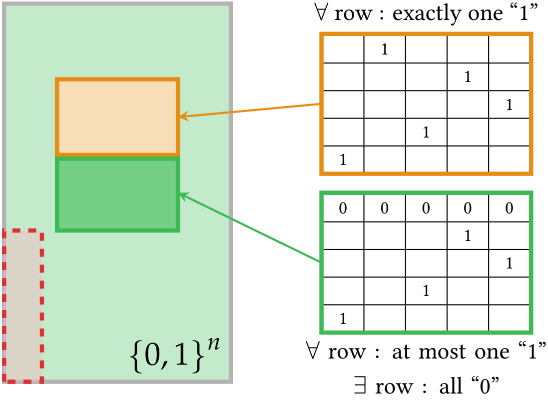

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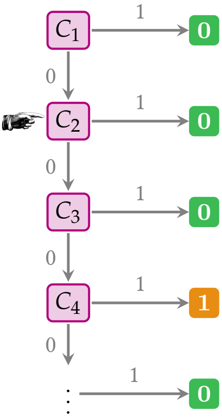

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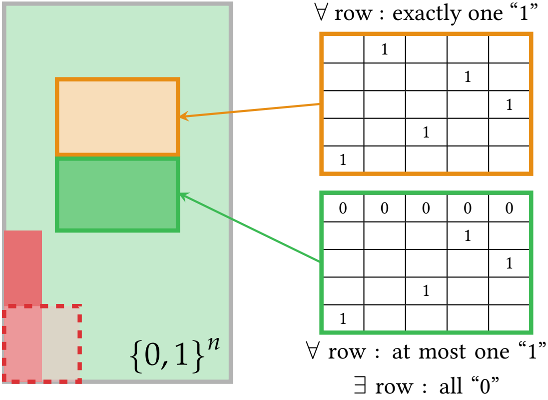

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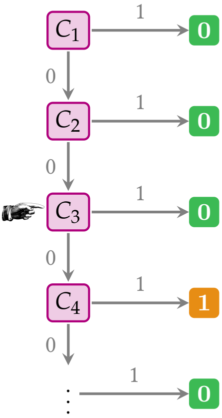

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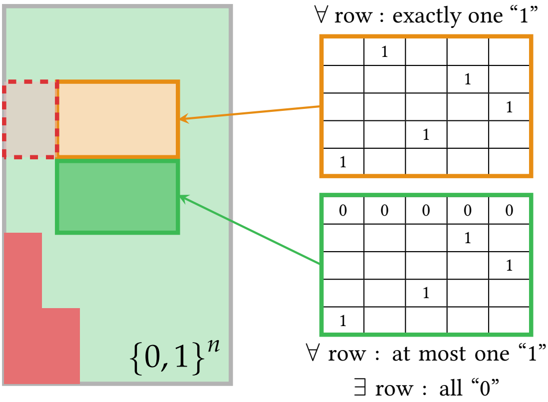

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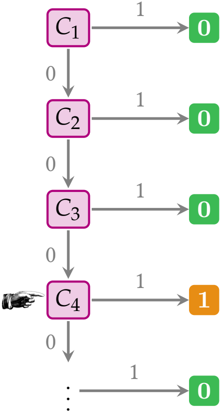

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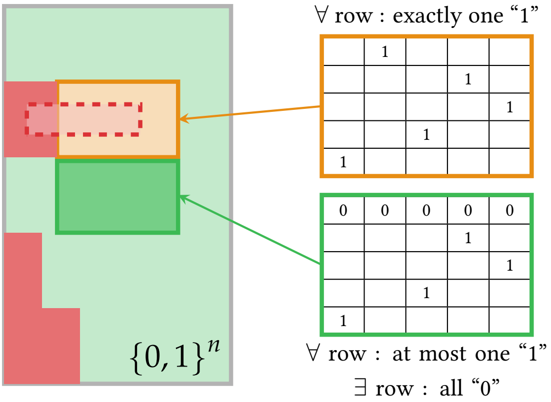

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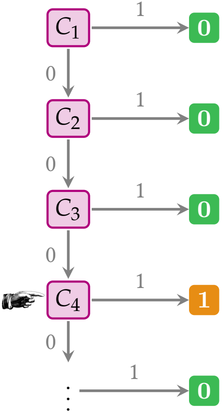
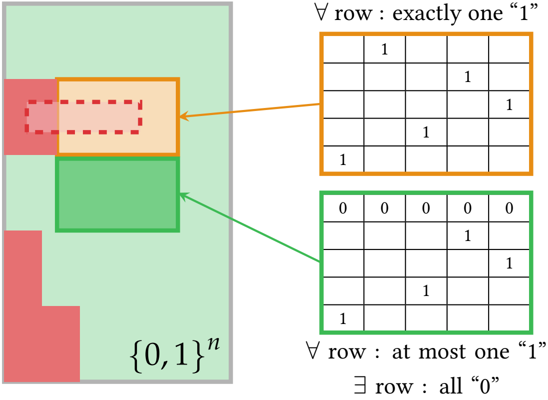


$\mathsf{width}(C_i) \leq k$

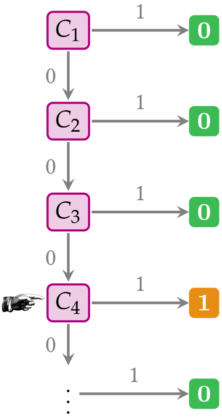
- C_i accepts some ■, but none in ■; or vice versa.



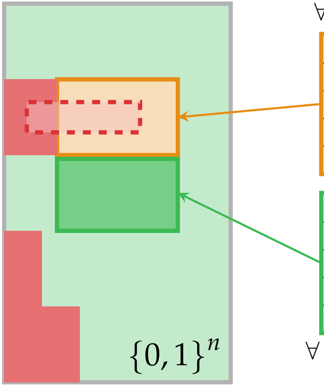
$\mathsf{DL}^{\mathrm{dt}}(f) \geq \sqrt{n}$

$$f(z) \overset{\text{def}}{=} \mathbb{1}\left\{\begin{array}{c}\text{Every row has} \\ \text{a unique “1”}\end{array}\right\}$$

- C_i accepts some ■ (orange), but none in ■ (green); or vice versa.
- C_i has to read at least one variable per row.



$\mathrm{width}(C_i) \leq k$



$\forall$ row : exactly one “1”

	1			
			1	
				1
		1		
1				

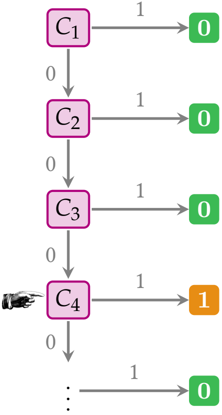
0	0	0	0	0
			1	
				1
		1		
1				

$\forall$ row : at most one “1”

$\exists$ row : all “0”

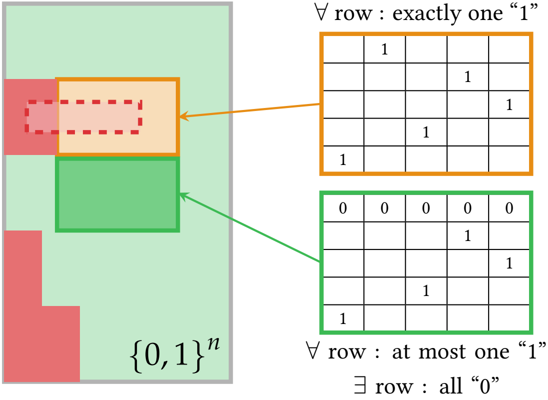
$\mathrm{DL}^{\mathrm{dt}}(f) \geq \sqrt{n}$

$$f(z) \stackrel{\text{def}}{=} \mathbb{1}\left\{\begin{array}{c}\text{Every row has}\\ \text{a unique "1"}\end{array}\right\}$$



$\mathrm{width}(C_i) \leq k$

- C_i accepts some ■ (orange), but none in ■ (green); or vice versa.
- C_i has to read at least one variable per row.
- That is, $\mathrm{width}(C_i) \geq \sqrt{n}$. □



Outline of Main Application

$$\mathsf{mon}(F) \overset{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \; \max\limits_{\text{m.c. } R} \; \mu(R)}$$

$$\exists\, F = \boxed{f} \circ \mathsf{Ind}^n_m$$

1

$$\boxed{\mathsf{mon}(F) \leq (\log n)^{O(1)}}$$

3

$$\mathsf{P}^{\mathsf{NPcc}}(F) \geq n^{\Omega(1)}$$

$\Updownarrow$ [PSS'14]

$$\mathsf{DL}^{\mathsf{cc}}(F) \geq n^{\Omega(1)}$$

$\Updownarrow$ [Lifting Theorem]

$$\boxed{\mathsf{DL}^{\mathsf{dt}}(f) \geq n^{\Omega(1)}}$$

2

Outline of Main Application

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$$\exists\, F = f \circ \mathsf{Ind}_m^n$$

✓

$$\mathsf{mon}(F) \le (\log n)^{O(1)}$$

3

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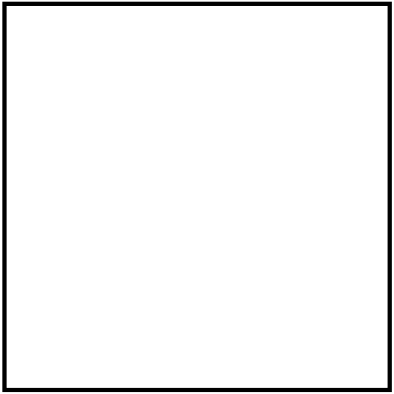
$$\mathsf{DL}^{\mathsf{cc}}(F) \ge n^{\Omega(1)}$$

$\Updownarrow$ [Lifting Theorem]

$$\mathsf{DL}^{\mathsf{dt}}(f) \ge n^{\Omega(1)}$$

✓

$\mathsf{mon}(F) \leq \mathsf{DL}^{\mathsf{cc}}(F)$ [IW10, PSS14]



$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \; \max\limits_{\text{m.c. } R} \; \mu(R)}$$

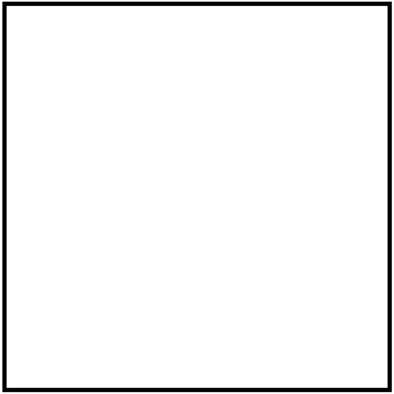
Given,

1. $\mathsf{DL}^{\mathsf{cc}}(F) = k$
2. Product measure μ

Find rectangle R, such that,

$$\mu(R) > 2^{-O(k)}$$

$\mathsf{mon}(F) \leq \mathsf{DL}^{\mathsf{cc}}(F)$ [IW10, PSS14]



$R_1 \quad R_2 \quad R_3 \quad \cdots \quad R_{2^k} \quad \top$

$$\mathsf{mon}(F) \stackrel{\text{def}}{=} \log \frac{1}{\min\limits_{\text{product } \mu} \max\limits_{\text{m.c. } R} \mu(R)}$$

Given,

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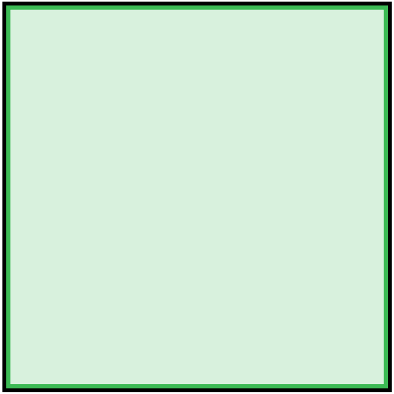
$\mathsf{mon}(F) \leq \mathsf{DL}^{\mathsf{cc}}(F)$ [IW10, PSS14]



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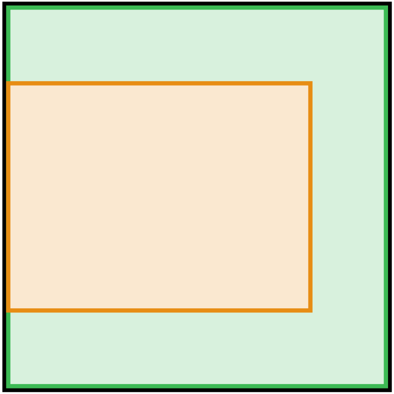

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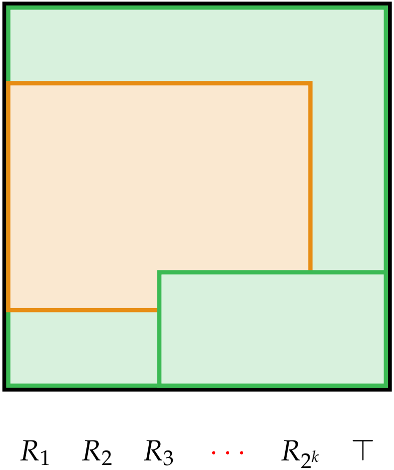

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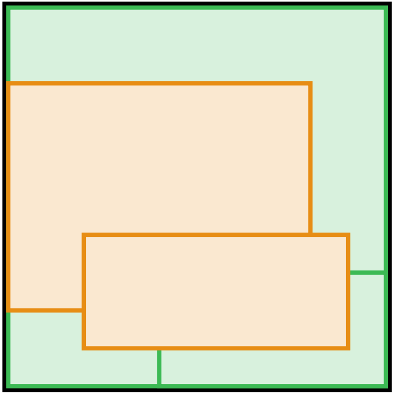

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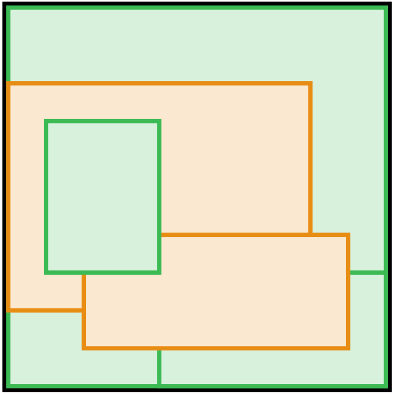

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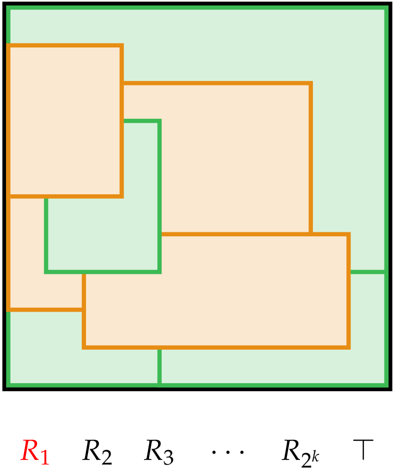

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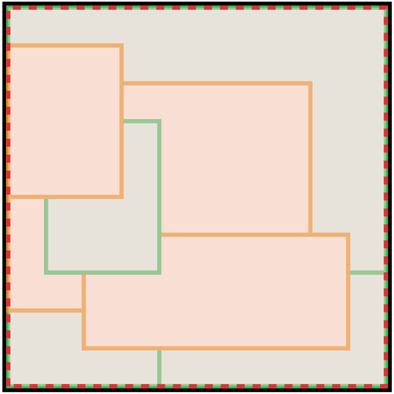

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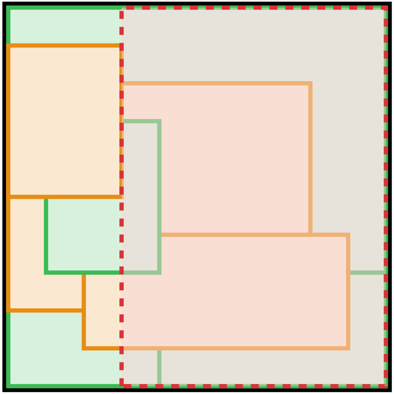

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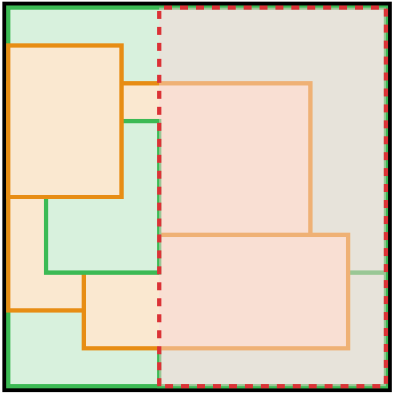

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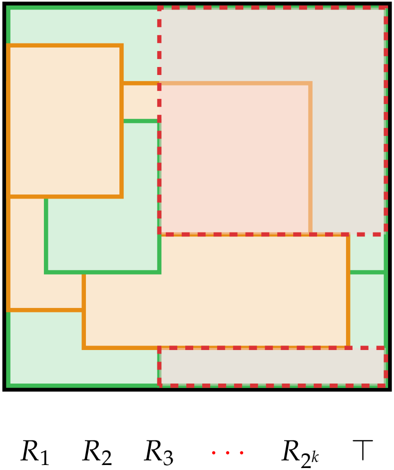

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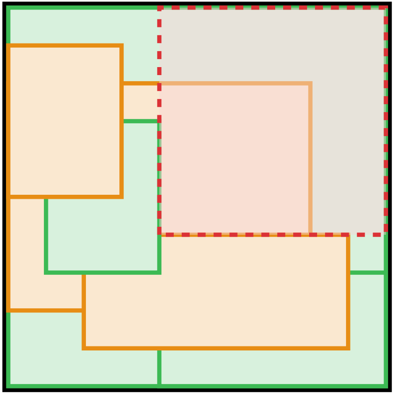

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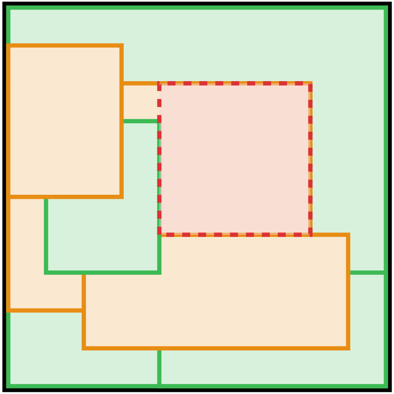

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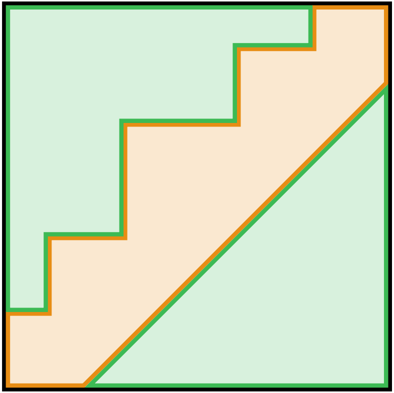
$$F = f \circ g^n$$

$$F \in \forall \cdot \mathsf{US}^{\mathsf{cc}}$$



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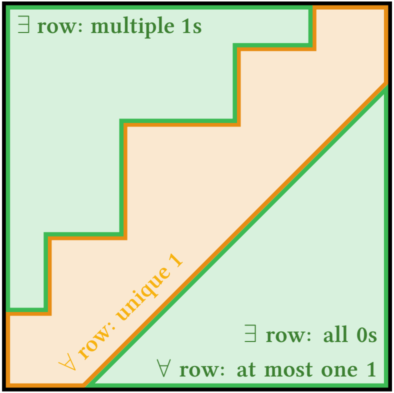

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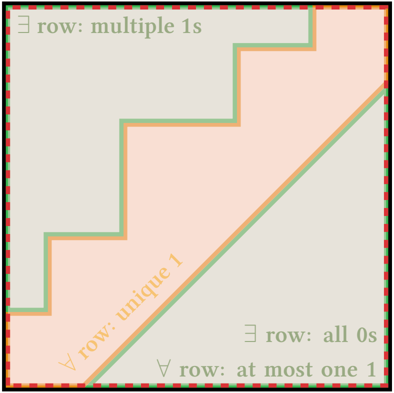

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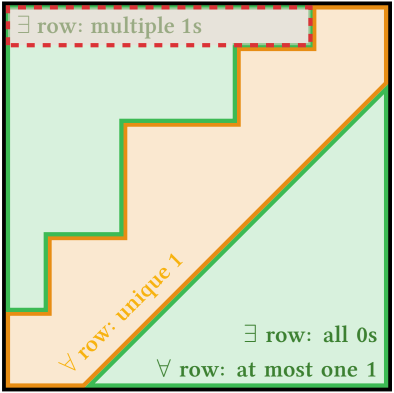

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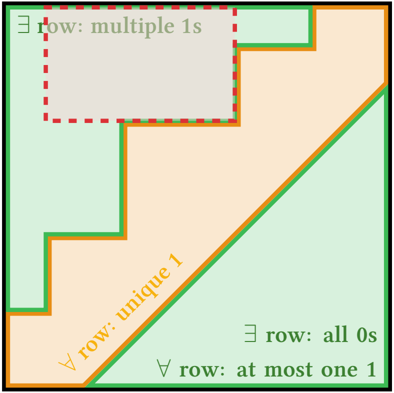

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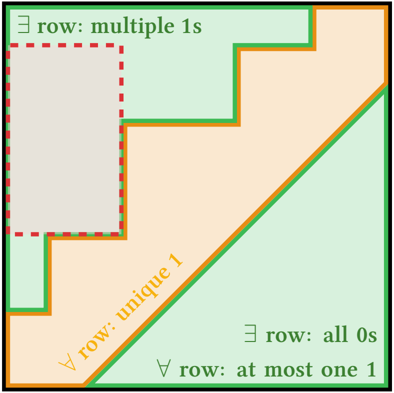

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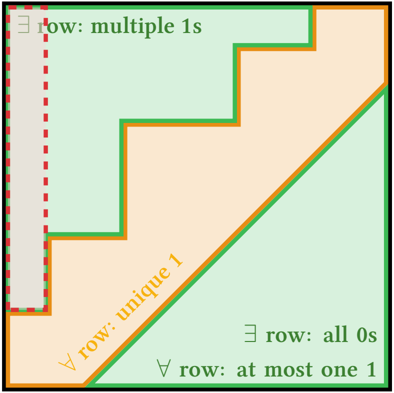

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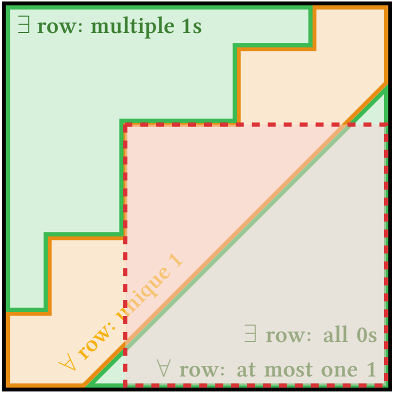

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[IW10, PSS14]

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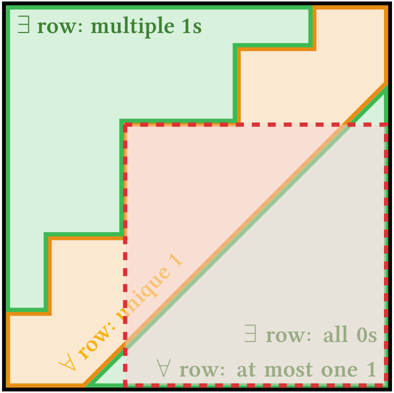

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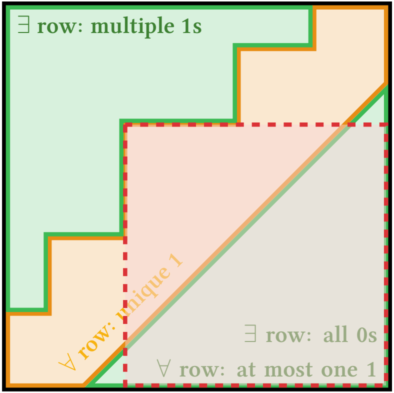

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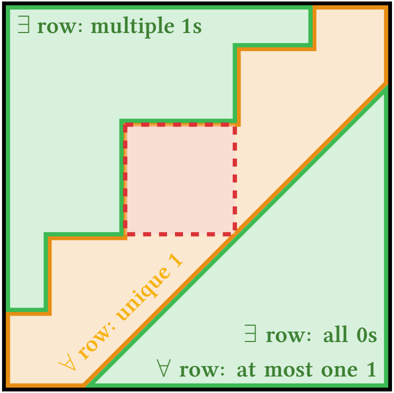

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done!

Summary & Open Questions

$$\exists\, F = f \circ \mathsf{Ind}_m^n$$

✓

$$\mathsf{mon}(F) \le (\log n)^{O(1)}$$

3

$$\mathsf{P}^{\mathsf{NPcc}}(F) \ge n^{\Omega(1)}$$

$\Updownarrow$ [PSS'14]

$$\mathsf{DL}^{\mathsf{cc}}(F) \ge n^{\Omega(1)}$$

$\Updownarrow$ [Lifting Theorem]

$$\mathsf{DL}^{\mathsf{dt}}(f) \ge n^{\Omega(1)}$$

✓

Summary & Open Questions

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Summary & Open Questions

Main Application

$$\exists F \quad : \quad \mathsf{mon}(F) \;\leq\; (\log n)^{O(1)}$$
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i.e. refutes:

$$\forall F \;:\; \mathsf{P}^{\mathsf{NPcc}}(F) \leq \mathsf{mon}(F)^{O(1)}$$

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Main Technique: Lifting Theorem for DL & P^{NP}

$$\mathsf{DL}^{\mathsf{cc}}(f \circ g^n) \geq \mathsf{DL}^{\mathsf{dt}}(f) \cdot \Omega(\log n)$$

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- Dual characterization of $\mathsf{DL}^{\mathsf{dt}}$
- [PSS14]-type method: Remove some rows AND some columns.
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Summary & Open Questions

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New! $\mathsf{BPP}^{\mathsf{cc}}(f \circ g^n) \geq \mathsf{BPP}^{\mathsf{dt}}(f) \cdot \Omega(\log n)$ [GPW17]

Summary & Open Questions

Open Questions

- $\exists F : \mathsf{mon}(F) \lll \mathsf{UPP}^{\mathsf{cc}}(F)$?
- Lifting with **constant-sized** gadgets?

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