Counting Basic-Irreducible Factors Mod p^k in Deterministic Poly-Time and p-Adic Applications

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Overview

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The lifting goes on same way for any power 3^k.

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It becomes non-trivial to find or even count all the factors.

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A completely unrelated irreducible factorization:

$f = (x + 351)\,(x + 135)\,(x^2 + 243x + 249) \bmod 3^6$.

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Extension to count irreducible factors will give an irreducibility criteria.

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Our result extends to count exactly the basic-irreducible factors of $f \bmod p^k$ as well.

Efficiently Partitioning the Root Set

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We give a simple exposition of [BLQ 13] which helps understand our deterministic algorithm.

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r is a lift of r_0 mod p, $r_0 + pr_1$ mod p^2 and so on.

Idea: Find each r_i one by one using the CZ algorithm to incrementally build up the lifts of r_0 with higher and higher precision leading up to r.

Randomized Algorithm: Notation

If $p^{\alpha}|f(x) \bmod p^k$ then any root $r = r_0 + pr_1 + \ldots + p^{k-1}r_{k-1}$ is independent of $r_{k-\alpha}$ and beyond.

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$r = r_0 + pr_1 + \ldots + p^{k-\alpha-1}r_{k-\alpha-1} + p^{k-\alpha} * + \ldots + p^{k-1}*$,
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where r is called a **representative root** representing p^α 'distinct' roots of f mod p^k.

The randomized algorithm will return **all** the roots in representative form- at most $\deg(f)$ many!

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Repeat the Shift-Divide cycle on $g(x) \bmod p^{k-\alpha}$ to get corresponding r_1s and so on.
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Always $\alpha \geq 1$, so the process stops in at most k iterations.

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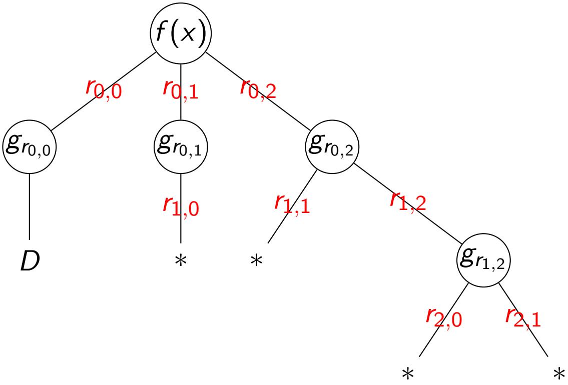

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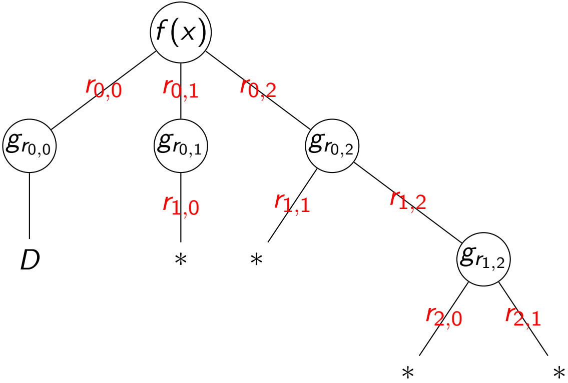

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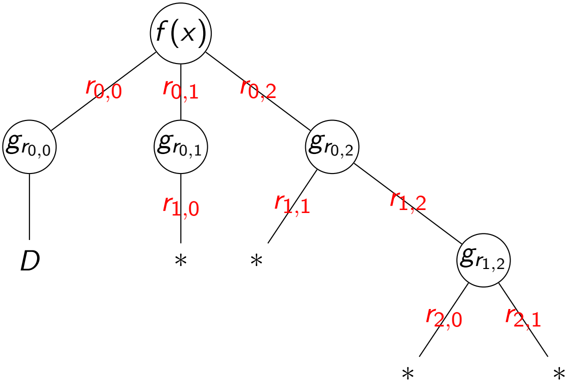

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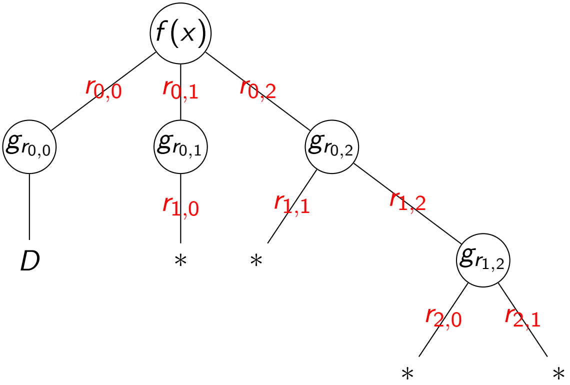

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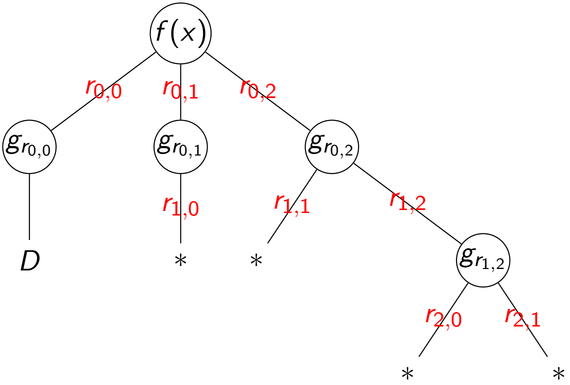

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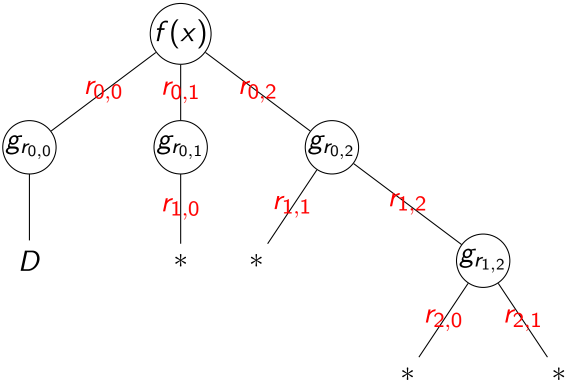

Partitioning the root-set: A path from root to a leaf denotes a representative-root of f. The tree has at most d leaves.

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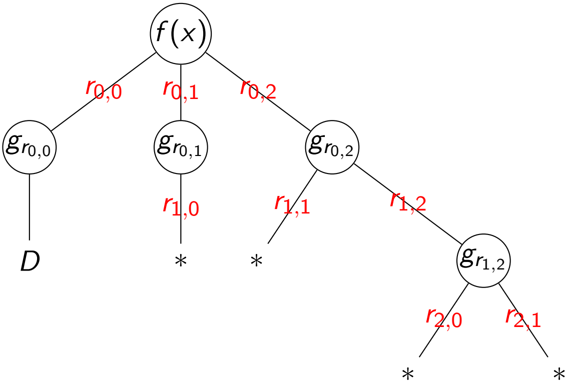

Claim: The degree of a node distributes to its children.

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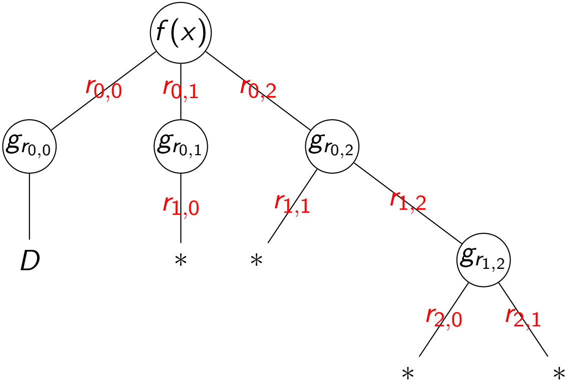

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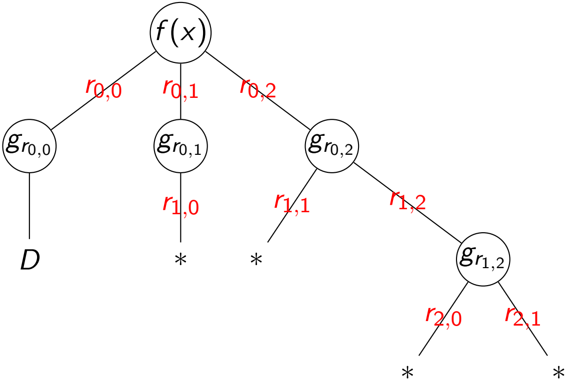

Multiplicity Property: Let r_0 be a root of multiplicity m of $f(x) \bmod p$ then the degree of children corresponding to r_0 is at most m.

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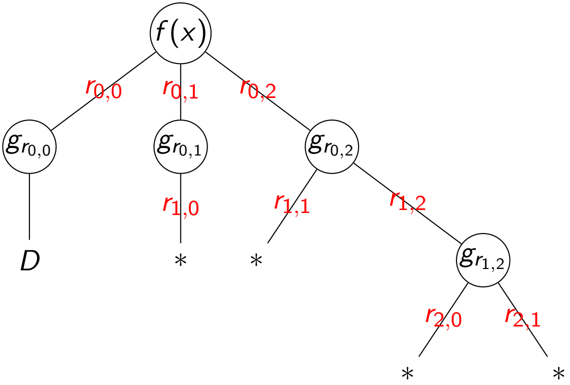

So, the size of tree is polynomial in input size and the algorithm runs in randomized poly($\deg(f)$, $k \log p$) time.

Overview

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Can we extend the techniques to count basic-irreducible factors $f \bmod p^k$?

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Last year Cheng, Gao, Rojas, Wan [ANTS' 18] partially derandomized in time exponential in the parameter k.

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We give the first deterministic poly$(d, k \log p)$ time algorithm to count the roots. A complete derandomization.

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Needs a different perspective.

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So we consider the representation- $x \to x_0 + px_1 + \ldots + p^{k-1} x_{k-1}$.

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$h(x)$ implicitly stores all the roots of g. The degree of h gives count!

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In the end, all ideals implicitly store all the roots of $f \bmod p^k$.

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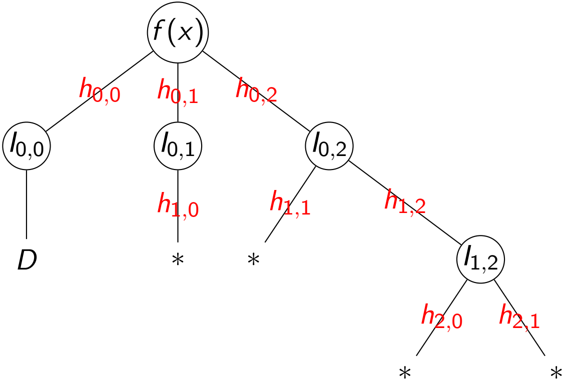

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Questions?

Thank You for your attention!