Constraints, Symmetry, and Complexity
Part 2

Andrei Krokhin

Mantra

- Symmetries of solution spaces are relevant for complexity
- Lack of symmetries $\Rightarrow$ hardness
- Symmetries $\Rightarrow$ efficient algorithms
- Symmetries of higher dimension/arity are important

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- k vs. $2^{\Omega(k^{1/3})}$ is **NP**-hard for large enough k [Huang'13]
- **NP**-hard for all $3 \leq k \leq c$, assuming non-standard (perfect completeness) variants of the UGC [Dinur, Mossel, Regev'09]

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3. Perhaps do both simultaneously — some time in the future

Promise CSP (PCSP)

Fix structures $\mathbb{A} = (A; R_1, \ldots, R_n)$ and $\mathbb{B} = (B; S_1, \ldots, S_n)$ s.t.

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Consider pairs of CSP instances:

$$\begin{aligned} \Phi_{strict} &= R_1(x,y,z), R_1(z,y,w), R_2(z), R_3(x,w), R_3(y,y) \\ \Phi_{relaxed} &= S_1(x,y,z), S_1(z,y,w), S_2(z), S_3(x,w), S_3(y,y) \end{aligned}$$

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Because of h, have Φ_{strict} is sat $\Rightarrow$ $\Phi_{relaxed}$ is sat

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Conjecture

The search and the decision problems are always equivalent.

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Theorem (Dinur, Regev, Smyth'05)

For any $2 \leq k \leq c$, PCSP(k-NAE, c-NAE) is **NP***-hard.*

Example: (1-in-3, NAE)-SAT

- 1-in-3 is $R_{1/3} = \{(0,0,1),(0,1,0),(1,0,0)\}$
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4. 1-in-3 instance sat $\Rightarrow$ lin system sat $\Rightarrow$ NAE instance sat

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Polymorphisms

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Can do polymorphisms from $\mathbb{A}$ to $\mathbb{B}$. Notation — $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$.
Example: n-ary functions from $\mathrm{Pol}(\mathbb{K}_k, \mathbb{K}_c) = c$-colourings of $\mathbb{K}_k^n$

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Can't compose polymorphisms!

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- There's a composition-free algebraic theory out there

Polymorphisms in control

Theorem (Pippenger'02, Brakensiek, Guruswami'18)

If $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{Pol}(\mathbb{C}, \mathbb{D})$ *then* $\mathrm{PCSP}(\mathbb{C}, \mathbb{D})$ *poly-time reduces to* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$.

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So this can take us some way, but we need a more abstract theory!

Minors and minions

Definition

Let $f : A^n \to B$ and let $\pi : [n] \to [m]$. We say that $g : A^m \to B$ is a minor of f (wrt. π) if $g(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

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Such sets are knowns as minions or clonoids.

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An identity of the form $g(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(m)})$ is called a minor identity.
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An identity of the form $g(x_1, \ldots, x_m) = f(x_{\pi(1)}, \ldots, x_{\pi(m)})$ is called a minor identity.
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Example:

$$\begin{aligned} s(x,y) &= f(x,x,y,y,y,x) \\ s(x,y) &= f(x,y,x,y,x,y) \\ s(x,y) &= f(y,x,x,x,y,y) \end{aligned}$$

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If every bipartite minor condition sat in $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ *is also sat in* $\mathrm{Pol}(\mathbb{C}, \mathbb{D})$ *then* $\mathrm{PCSP}(\mathbb{C}, \mathbb{D})$ *logspace reduces to* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$.

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bmc's sat in $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is the right notion of symmetry for PCSPs!

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- Both $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ and $\mathrm{Pol}(2 - NAE, 458 - NAE)$ satisfy many non-trivial bmc's

Proof

Lemma (Bulín, AK, Opršal)

To prove the theorem, it is enough to show that $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ *does* <u>not</u> *satisfy the following bmc:*

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Let's show that $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ contains no such f. What would such an f be? - a 5-colouring of $\mathbb{K}_3^6$ that that gives the same colour to some specified vertices.

End of proof

Take $\mathbb{K}_3^6$ and, for each pair $(x, y) \in [3]^2$, glue together 3 vertices: (x, x, y, y, y, x), (x, y, x, y, x, y), (y, x, x, x, y, y).

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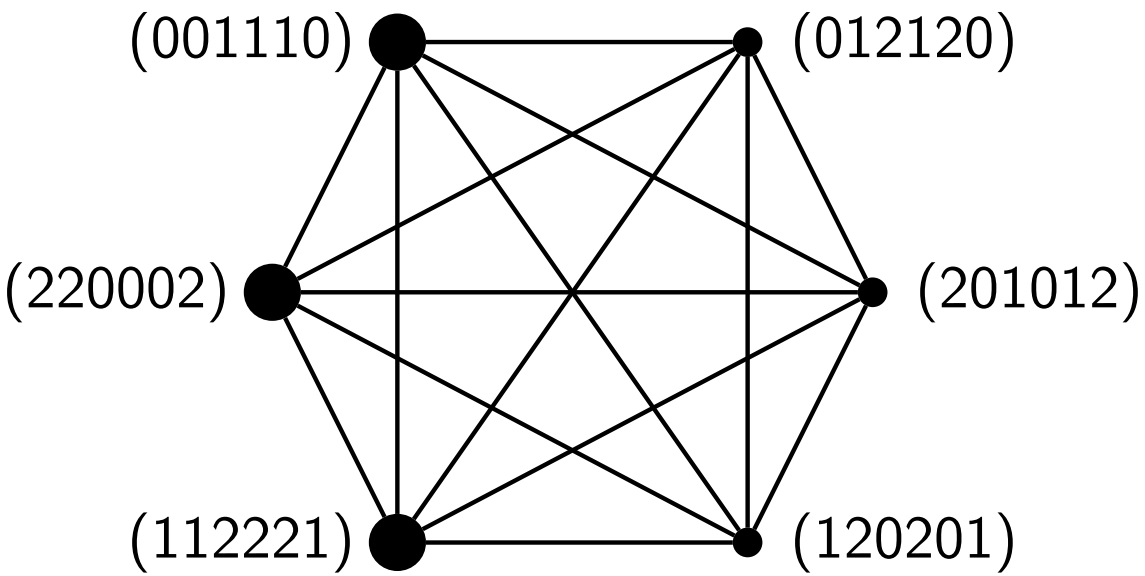
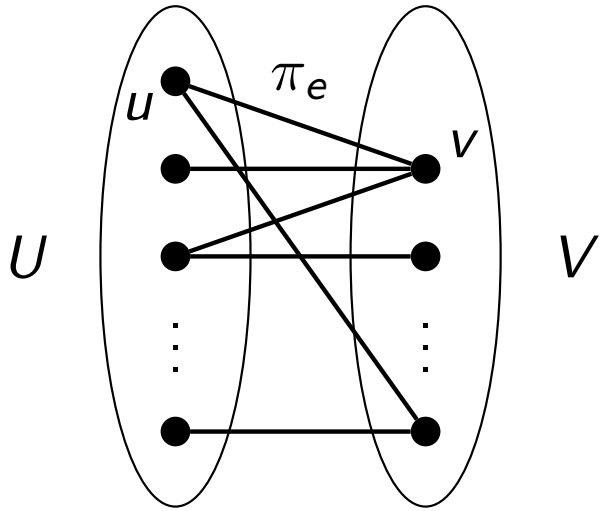


The Label Cover problem

The Label Cover, $\mathrm{LC}(N)$ problem is a CSP that looks like this:



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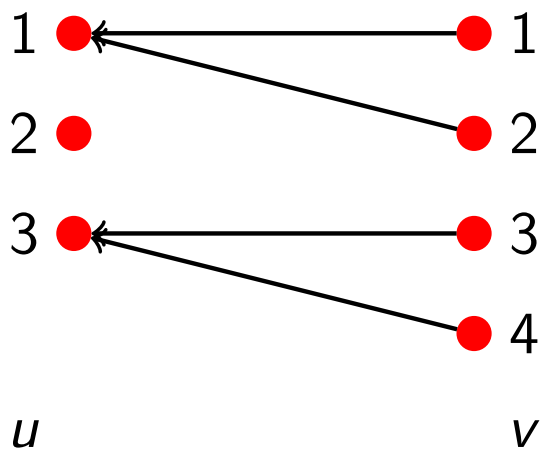

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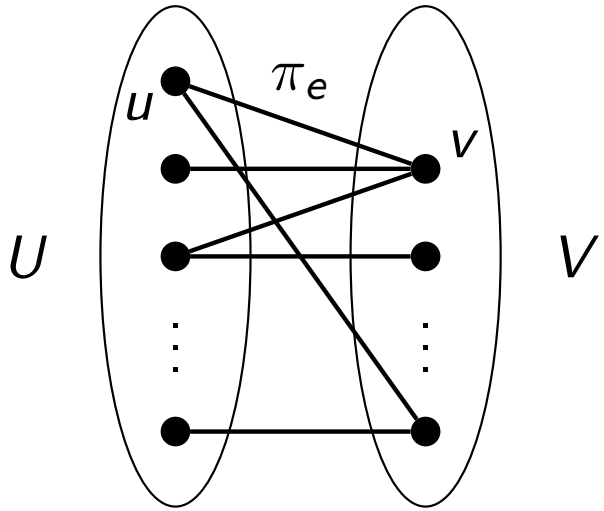

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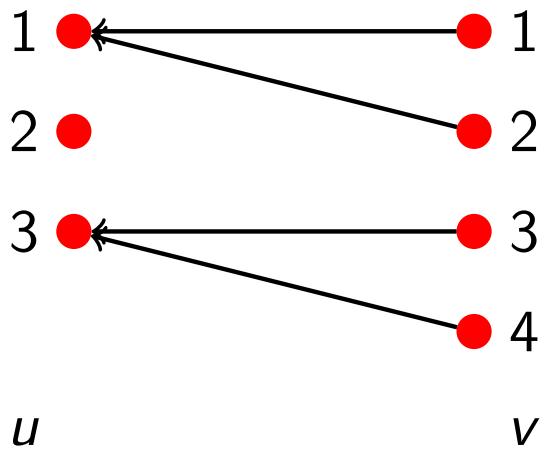

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Fact

For any $N \geq 3$, $\mathrm{LC}(N)$ is **NP***-complete.*

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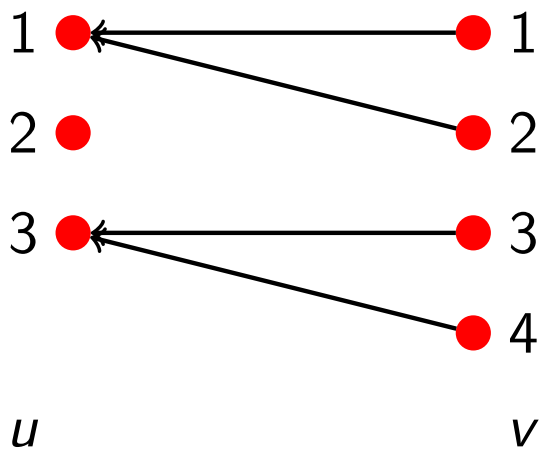

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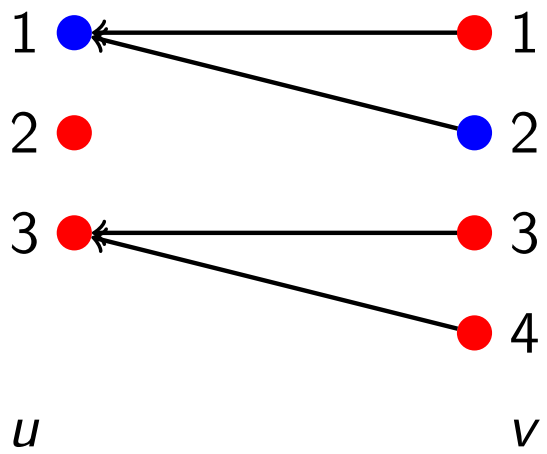

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Gap Label Cover (GLC)

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For $\epsilon > 0$ and integer N, the problem $\mathrm{GLC}_{1,\epsilon}(N)$ is, given an instance of $\mathrm{LC}(N)$, to distinguish between two cases:

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$\mathrm{GLC}_{1,\epsilon}(N)$ is equivalent to the problem of distiguishing whether a given bmc is trivial or any ϵ-fraction of it is non-trivial.

The Promise MC problem (=PCSP)

Definition

Fix a set $\mathcal{M}$ of functions. For an integer N, the problem $\mathrm{PMC}_{\mathcal{M}}(N)$ is, given a bmc, to distinguish between two cases:

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2. The bmc is sat in $\mathcal{M} \Rightarrow$ ϵ-fraction of the bmc is non-trivial.

   Since bmc $=$ LC, ϵ-fraction of the LC constraints are now sat.

Key idea: (2) should work because symmetries of $\mathcal{M}$ are limited.

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- Gap between "limited non-trivial" and "strong enough"?
- Need more evidence, e.g. classifications for
  - all Boolean PCSPs? (cf. [Schaefer'78] for CSP)
  - promise graph homomorphisms? (cf. [Hell-Nešetřil'90])

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