Constraints, Symmetry, and Complexity

Andrei Krokhin

Message

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- The Constraint Satisfaction Problem paradigm
  - General enough to include many interesting problems
  - Structured enough to make significant progress (on all problems within a class, not just a single problem)

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- The Constraint Satisfaction Problem paradigm
- Main Achievement: better understanding of common structure in easy (or hard) problems
  - — focus on appropriate symmetries of solution spaces!
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  - useful way to measure/compare symmetries
- Long term goal: go beyond CSPs

Constraint Satisfaction Problem (CSP)

Fix finite relational structure $\mathbb{A} = (A; R_1, \ldots, R_n)$ (aka constraint language) where each $R_i \subseteq A^{k_i}$ or $R_i : A^{k_i} \to \{\text{true}, \text{false}\}$.

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Definition

An instance of $\mathrm{CSP}(\mathbb{A})$ is a list of constraints over vars V, e.g.

$$R_1(x, y, z), R_1(z, y, w), R_2(z), R_3(x, w), R_3(y, y)$$

where each R_i is from $\mathbb{A}$.

Question: Is there $s : V \to A$ satisfying all constraints?

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Other variants, e.g.:

- infinite A (but the instance is still finite)
- nothing (or something other than relations) is fixed
- real-valued functions instead of relations (for optimisation)

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  - (essentially) k-colouring for 3-uniform hypergraphs

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Conjecture (CSP Dichotomy Conjecture, Feder-Vardi'98)

Every $\mathrm{CSP}(\mathbb{A})$ *is either in* **P** *or* **NP**-*c.*

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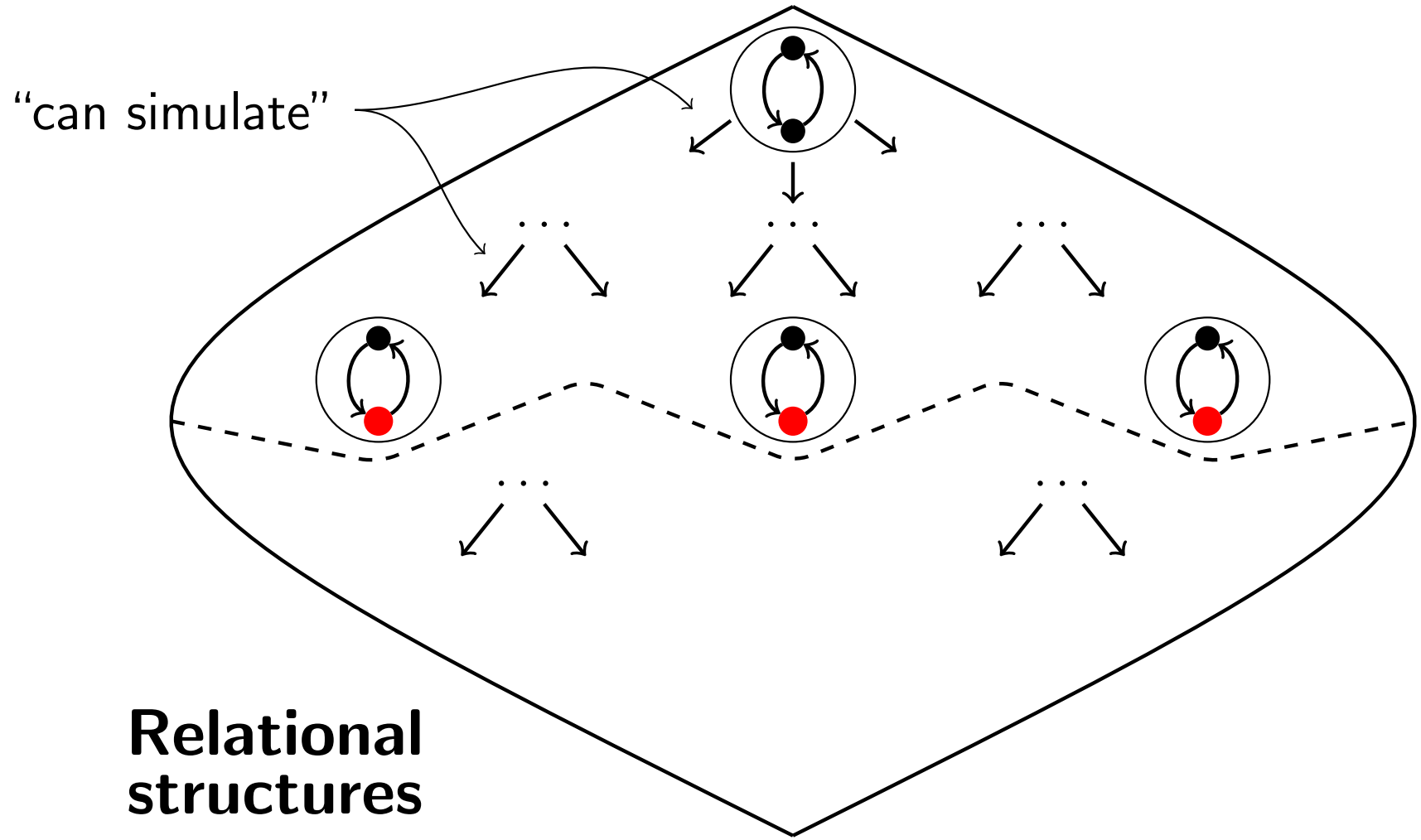
6. **Promise CSP**: wide open — tomorrow

Algebraic approach to CSP (very high-level view)



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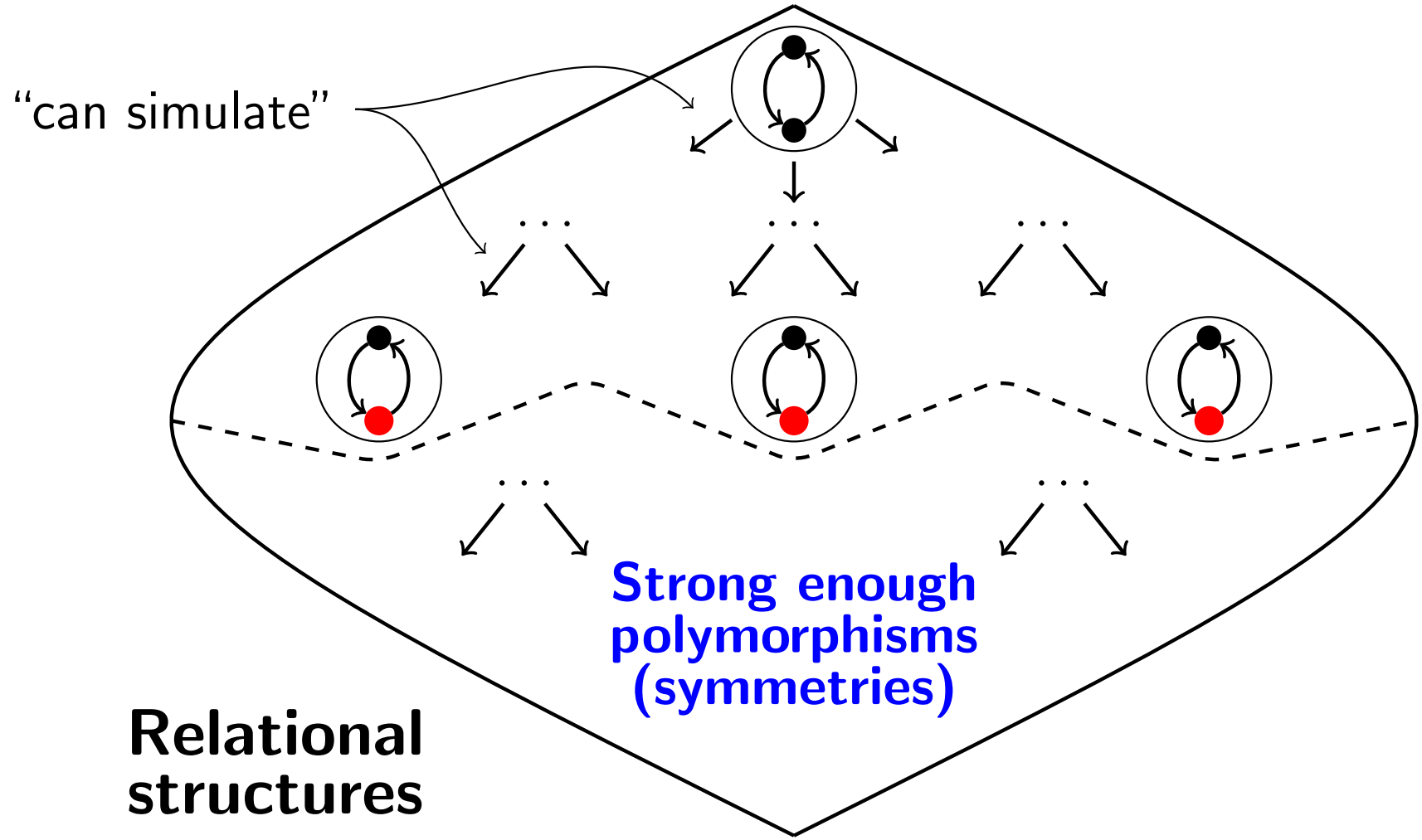

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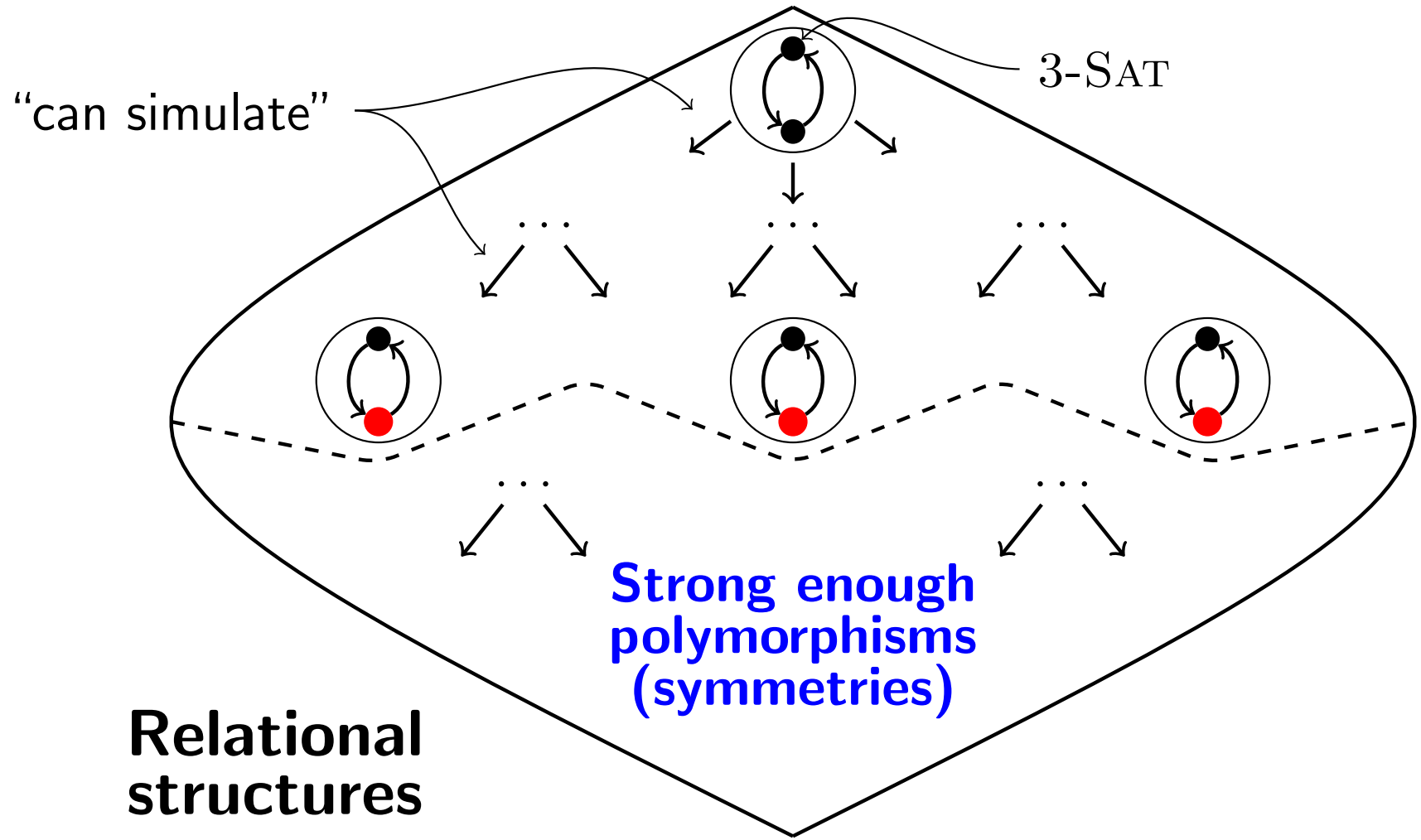

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2. pp-interpretations (gadgets, possibly diff domains)
3. pp-constructions (gadgets + more)

Polymorphisms by example

- Take any 2-SAT instance $(x \vee \overline{y}) \wedge (y \vee \overline{z}) \wedge (y \vee \overline{u}) \wedge (x \vee u)$
- Take any three solutions $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to this instance

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$$
\begin{array}{rcccccc}
 & & m & m & m & m & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbf{a} = & (& 1 & 1 & 1 & 0 &) \quad \text{sat} \\
\mathbf{b} = & (& 1 & 1 & 0 & 1 &) \quad \text{sat} \\
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\hline
m(\mathbf{a}, \mathbf{b}, \mathbf{c}) = & (& 1 & 1 & 0 & 0 &) \quad \text{sat}
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Polymorphisms

An operation $f : A^m \to A$ is called a polymorphism of a k-ary relation $R \subseteq A^k$ if for any k-tuples

$$\begin{array}{ccccccc}
 & f & & f & & f & \\
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Call f a polymorphism of $\mathbb{A}$ if it is such for all R in $\mathbb{A}$.
Notation: $\mathrm{Pol}(\mathbb{A})$.

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4. Every polymorphism of 3-COL is of the form
   $f(x_1, \ldots, x_n) = \pi(x_i)$ for some $i \leq n$ and permutation π

Modern proof of Schaefer's classification for Boolean CSPs

Problem	Polymorphism	Complexity
0-valid	constant 0	Easy
1-valid	constant 1	Easy
2-SAT	majority	Easy
HORN-SAT	min	Easy
DUAL H-SAT	max	Easy
3-LIN2	$x - y + z$	Easy
3-SAT	only projections	Hard

Polymorphisms as symmetries

Objects capturing the symmetry of $\mathrm{CSP}(\mathbb{A})$:

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- $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$ - no info about the complexity of $\mathrm{CSP}(\mathbb{A})$.
- $\mathrm{Pol}(\mathbb{A})$ - a lot of info.

Simulation vs. polymorphisms

Theorem (Birkhoff'35; Geiger'68; Bodnarchuk et al.'69; Bodirsky; Willard; Barto, Opršal,Pinsker'18)

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  - Not used in resolving Dichotomy Conj, but very important

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  - $f(x_1,x_2,x_3) = f(x_1,x_3,x_2)$ trivial symmetry
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Systems of (h1) identites sat in $\mathrm{Pol}(\mathbb{A})$ = measure of symmetry.

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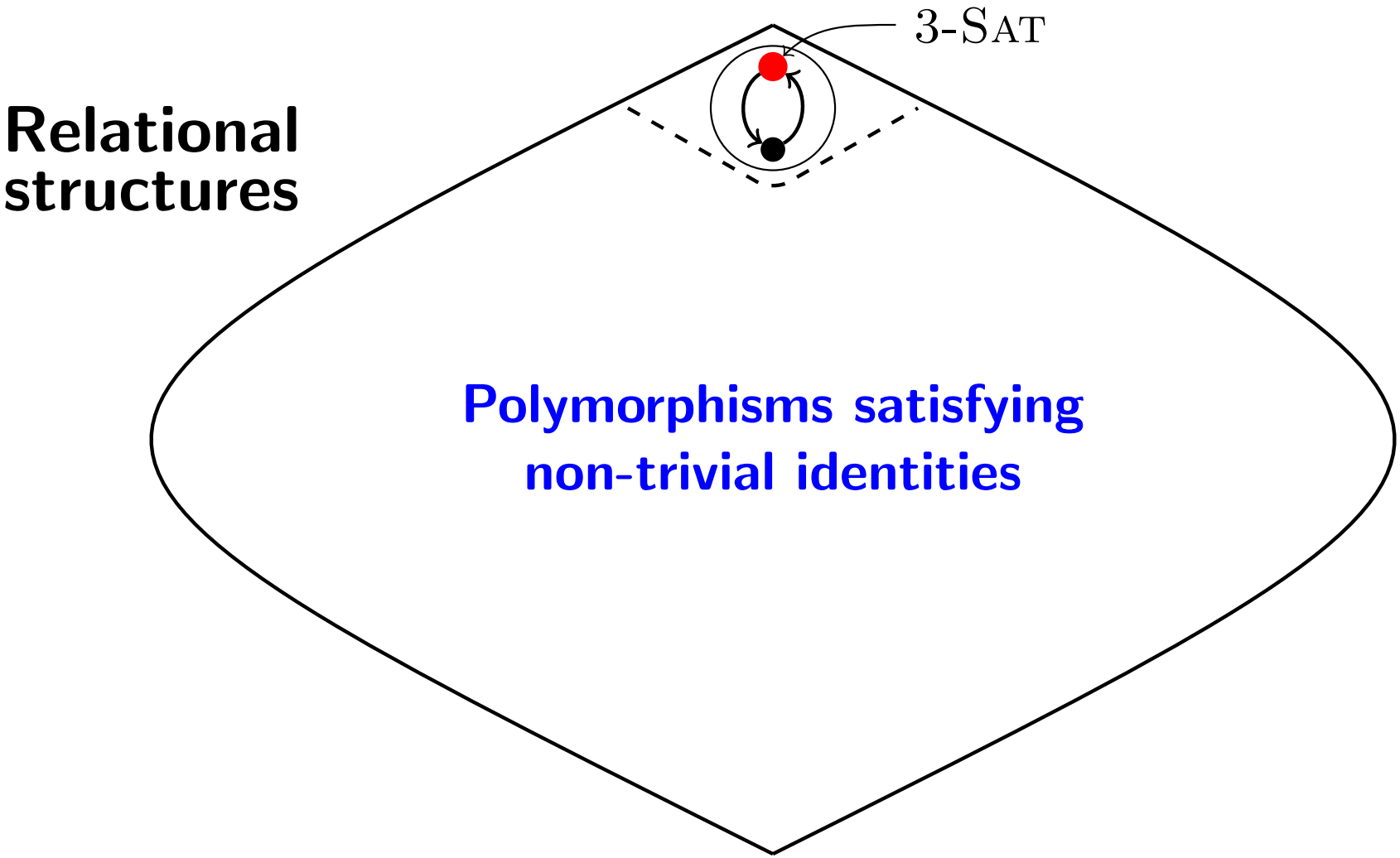
- If $\mathrm{Pol}(\mathbb{A}) \not\to \mathcal{T}$, $\mathrm{Pol}(\mathbb{A})$ satisfies *some* non-trivial identities.

Algebraic dichotomy (picture)



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Theorem (Idziak, Markovic, McKenzie, Valeriote, Willard'10)

Certain identities sat in $\mathrm{Pol}(\mathbb{A}) \Leftrightarrow$ *(a). If have (a), can do (b).*

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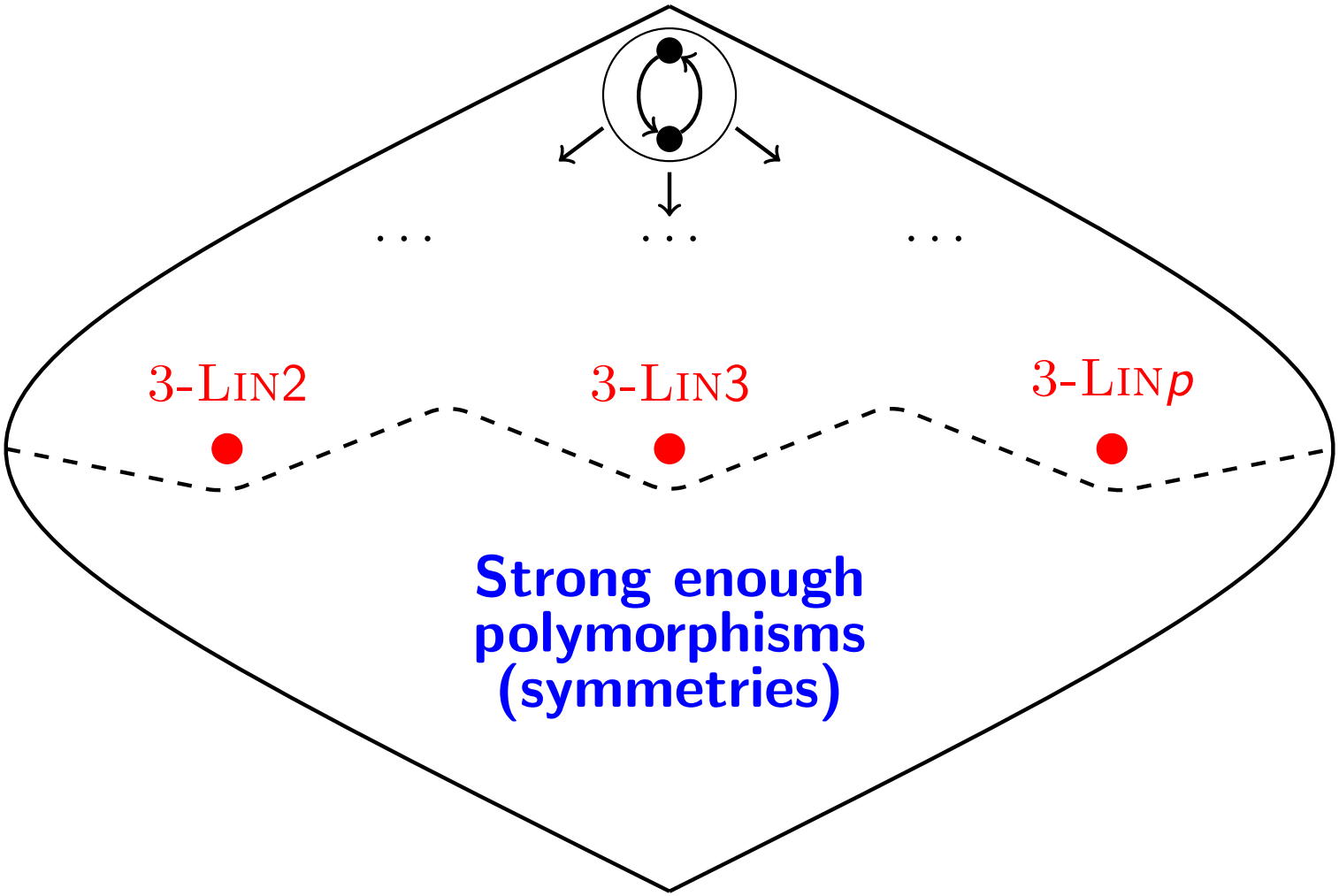
Equivalent notions: treewidth duality, Datalog, pebble games, etc

Picture for bounded width?



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Polymorphisms are used only to prove correctness of the algorithm.

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Theorem (Bulatov'17, Zhuk'17)

If $\mathbb{A}$ *has a weak near-unanimity polymorphism (equivalently,* $\mathrm{Pol}(\mathbb{A}) \not\to \mathcal{T}$*) then* $\mathrm{CSP}(\mathbb{A})$ *is in* **P**.

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- if some $dom(v)$ can be reduced without losing all solutions, reduce it and recurse
- if an instance can be split/decomposed into a constant number of disjoint instances, split/decompose and recurse
- this can be done in many situations

Bulatov's and Zhuk's algorithms (very roughly), 2

The base cases for recursion (roughly) are:

- (B and Z) all domains 1-element $\rightarrow$ solution
- (B and Z) some domain empty $\rightarrow$ no solution

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- (B and Z) all domains 1-element $\rightarrow$ solution
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- (B) few subpowers algorithm is applicable
- (Z) systems of linear equations over $\mathbb{Z}_p$

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However, the recursion is organised very differently:

- (B) heavily uses deep algebraic structure
- (Z) still very algebraic, but more elementary
- (B and Z) a whole range of diff algebraic tricks

Feder-Vardi conjecture proved – End of the road ???

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Actually, we are rather closer to the beginning!

Message

1. **Decision CSP**: Can all constraints be satisfied?
2. **Counting CSP**: Count the number of solutions
3. **Max CSP**: Find a map satisfying max number of constraints
4. **Approx Max CSP**: Satisfy $c \times \mathrm{Opt}$ number of constraints
5. **Approx Min CSP**: assuming $1 - \epsilon$ fraction of constraints can be satisfied, find a map satisfying $\geq 1 - g(\epsilon)$ fraction.
6. **Promise CSP**: ... tomorrow

Each of the above has an appropriate notion of symmetry!

- lack of symmetries $\Rightarrow$ hardness
- symmetries $\Rightarrow$ efficient algorithms
- symmetries of higher dimension/arity are important

Useful surveys

Much of what is covered in this lecture can be found in survey

- Polymorphisms, and how to use them.
  L. Barto, A. Krokhin, and R. Willard.

It is written specifically for those without algebra background!

See also the full (open-access) volume of surveys:

- The Constraint Satisfaction Problem: Complexity and Approximability. Editors: A. Krokhin and S. Živný. Dagstuhl Follow-Ups series, Volume 7, 2017.
  http://drops.dagstuhl.de/portals/dfu/index.php?semnr=16027