Communication Complexity with Small Advantage

Thomas Watson

University of Memphis

Communication complexity

$$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

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(Alice: x) (Bob: y)

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$\longleftarrow$

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$$\text{(Alice: } x) \qquad \xrightarrow{\hspace{3em}} \qquad \text{(Bob: } y)$$

$$\xleftarrow{\hspace{3em}}$$

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$F(x,y)$ $F(x,y)$

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Randomized protocols:

Communication complexity

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$$\begin{array}{ccc} \text{(Alice: } x\text{)} & \longrightarrow & \text{(Bob: } y\text{)} \\ & \longleftarrow & \\ & \longrightarrow & \\ & \longleftarrow & \\ & \longrightarrow & \\ F(x,y) & & F(x,y) \end{array}$$

Randomized protocols:

- Correctness: $\forall (x,y) :\ \mathbb{P}[\text{output is } F(x,y)] \geq 3/4$

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- Correctness: $\forall (x,y) :\ \mathbb{P}[\text{output is } F(x,y)] \geq 3/4$
- Cost: $\max_{(x,y),\ \text{random outcomes}}(\#\text{ bits communicated})$
- Complexity: $\mathrm{R}(F) = \min_{\text{correct protocols}}(\text{cost})$

Classic results

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$$\begin{aligned}
\mathsf{R}(\textsc{Inner-Product}) &= \Theta(n) \\
\mathsf{R}(\textsc{Set-Intersection}) &= \Theta(n) \\
\mathsf{R}(\textsc{Gap-Hamming}) &= \Theta(n)
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R : success probability $\geq 3/4$

Small advantage: $\mathsf{R}_{1/2+\epsilon}$: success probability $\geq 1/2 + \epsilon$

Classic results — revisited

$$
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\mathsf{R}_{1/2+\epsilon}(\textsc{Inner-Product}) &= \Theta(n) \qquad (\text{unless } \epsilon \leq 2^{-\Omega(n)}) \\
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⋮

other functions?

Climbing the polynomial hierarchy

NP:

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[Braverman–Moitra STOC'13, Göös–Watson RANDOM'14]
(information complexity) (corruption)

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Σ_2P, Π_2P:

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Constant advantage: well-understood
[Jayram–Kopparty–Raghavendra/Leonardos–Saks CCC'09]

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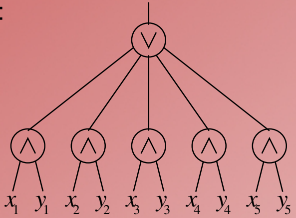
Constant advantage: well-understood
[Jayram–Kopparty–Raghavendra/Leonardos–Saks CCC'09]

Small advantage: open

Function definitions

SET-INTERSECTION:



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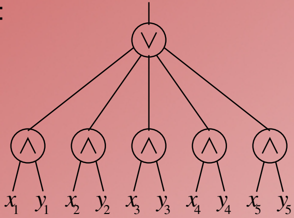
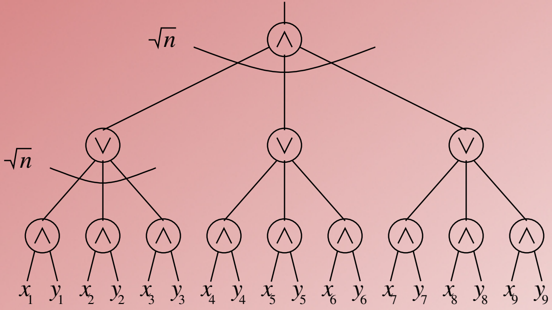


TRIBES:



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- Not true in general

Our approach for TRIBES

Information complexity:

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- $\Omega(1)$-advantage for TRIBES [JKS'03]

Our approach for TRIBES

Information complexity:

- $\Omega(1)$-advantage for TRIBES [JKS'03]
- ϵ-advantage for SET-INTER [BM'13]

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- Combine?

Our approach for Tribes

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- $\Omega(1)$-advantage for Tribes [JKS'03]
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4-step approach:

Our approach for TRIBES

Information complexity:

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4-step approach:

1. Conditioning and direct sum
2. Uniformly covering a pair of gadgets
3. Relating information and probabilities for inputs
4. Relating information and probabilities for transcripts

Preliminaries

Idea from [BM'13]:

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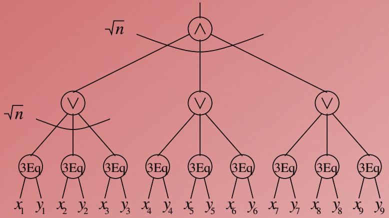
Suffices to use $3\textsc{Eq}$ gadget instead of $\textsc{And}$ gadget

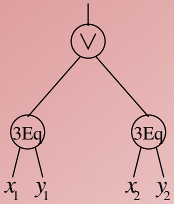
$$3\textsc{Eq}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textsc{And}: \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1. Conditioning and direct sum

info cost $o(\epsilon \cdot n)$ $\rightsquigarrow$ info cost $o(\epsilon)$



$\rightsquigarrow$



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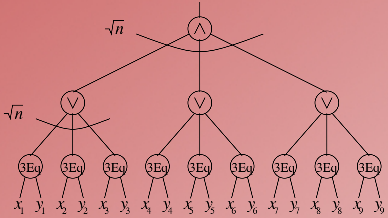

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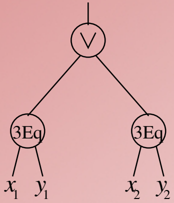

Want to show:

advantage $\leq O$(info cost)

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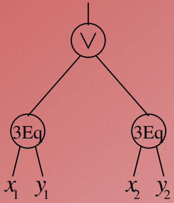

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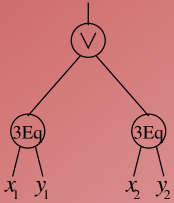

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Usual info complexity proofs use Pinsker:

statistical distance $\leq O(\sqrt{\text{mutual info}})$

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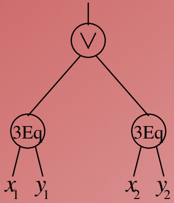

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Usual info complexity proofs use Pinsker:

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Instead: exploit symmetry properties of $3\mathrm{EQ}$ to get higher-order terms to "cancel out"

2. Uniformly covering a pair of gadgets

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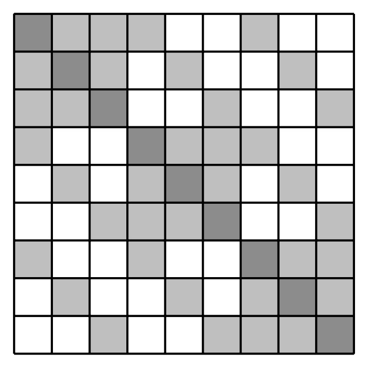
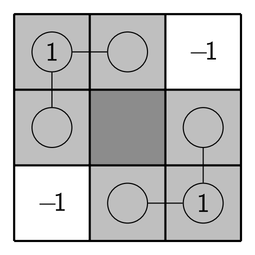
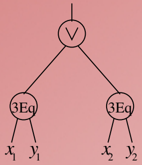
Uniformly cover



with





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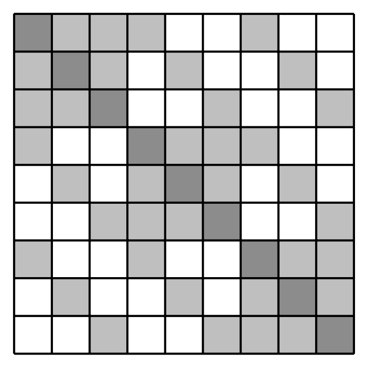
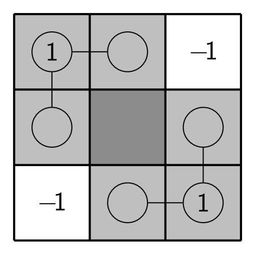
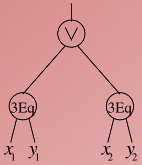
Uniformly cover  with 



Lemma: Linear combination of acceptance probabilities
$\leq O(\sum$ four contributions to info cost$)$

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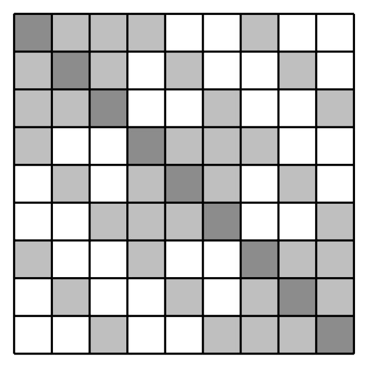

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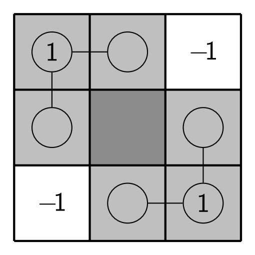

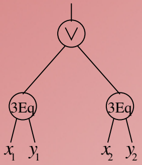

Lemma: Linear combination of acceptance probabilities
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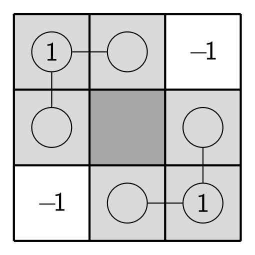
Uniform covering + Lemma:
advantage $\leq O($info cost$)$

3. Relating information and probabilities for inputs



"Input Lemma":

Linear combination of acceptance probabilities
$\leq O(\sum$ four contributions to info cost$)$

Prove analogous "Transcript Lemma"?

(then could sum over transcripts)

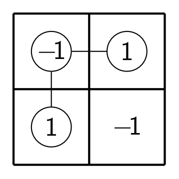
3. Relating information and probabilities for inputs

[BM'13] transcript lemma:

Our transcript lemma:

3. Relating information and probabilities for inputs

[BM'13] transcript lemma:



Our transcript lemma:

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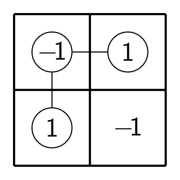

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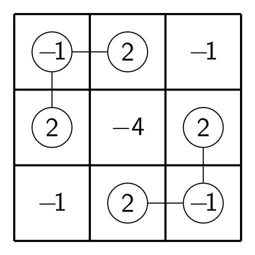

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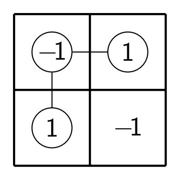

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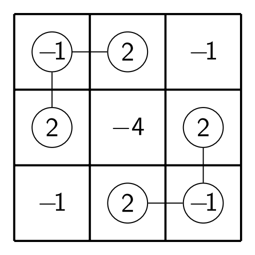

$\forall$ transcript:

contribution to lin comb of prob $\leq$ O(contribution to info costs)

3. Relating information and probabilities for inputs

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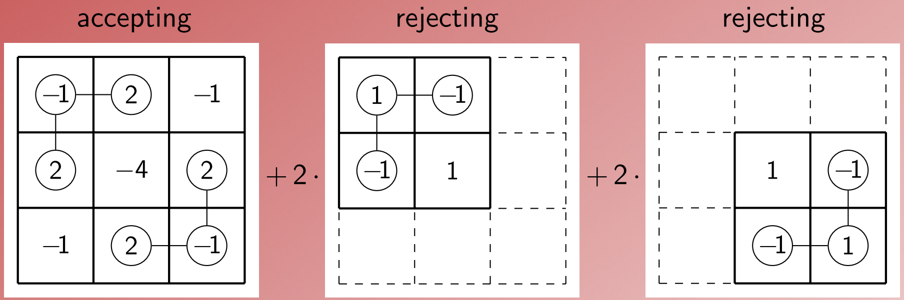
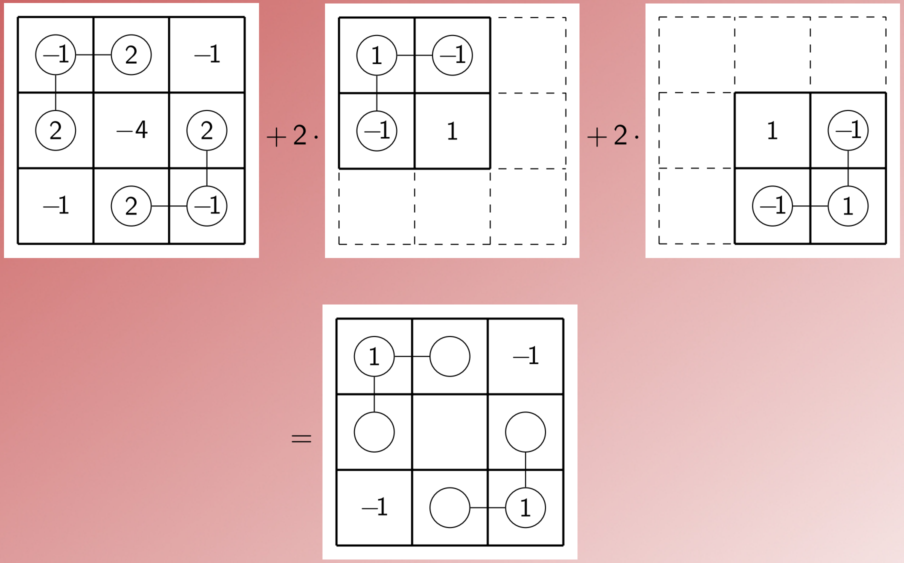
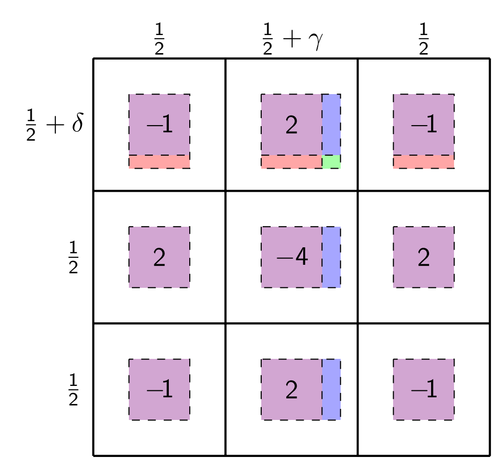


3. Relating information and probabilities for inputs

accepting rejecting rejecting



4. Relating information and probabilities for transcripts



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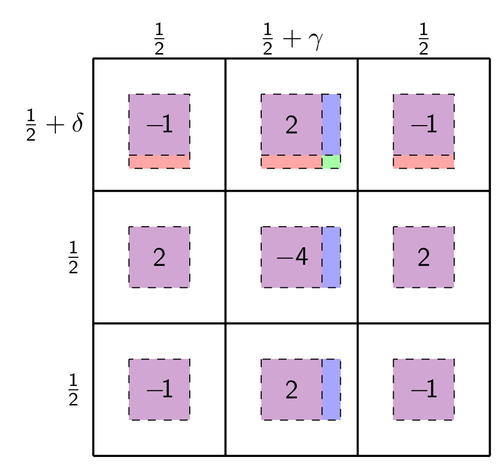
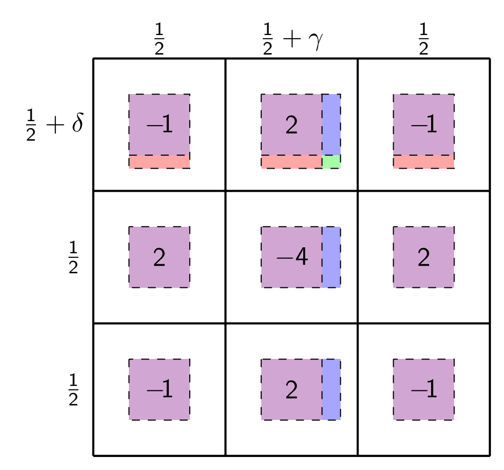


lin comb of probabilities
$= 2 \cdot$ green area
$= \Theta(\delta\gamma)$

4. Relating information and probabilities for transcripts



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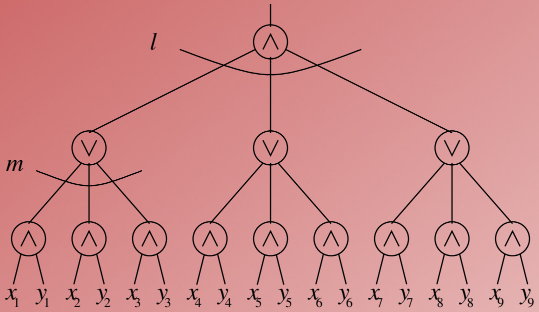
$\leq$

contribution to info costs
$= \Theta(\delta^2 + \gamma^2)$

Generalized TRIBES



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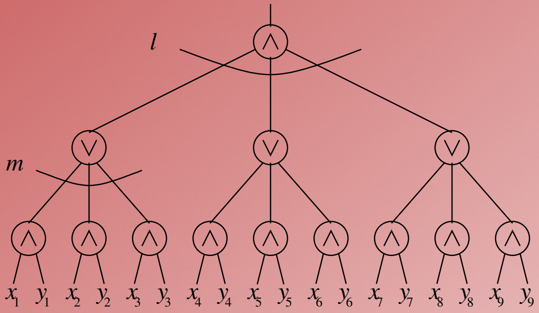

Open:

Small-advantage complexity when ℓ is small

The end