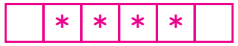
Hardness of Function Composition for Semantic Read once Branching Programs

Jeff Edmonds, Venkatesh Medabalimi, Toniann Pitassi

June 23, 2018

P AND L

P: Polynomial time computable functions.



L : Functions computable in logarithmic space.

$$L \overset{?}{\subset} \mathbf{P}$$

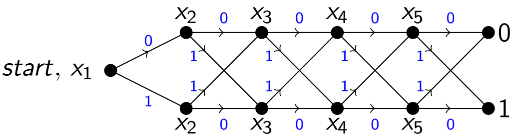
Branching Programs

$f(x_1, x_2, .., x_n) \to \{0,1\}$ $\qquad$ $x_i \in \{0,1\}, \forall i \in [n]$

Definition

Deterministic Branching program

- DAG with a source node and two sinks, 1-sink (for accept) and 0-sink (for reject).
- Each non-sink node is labeled by some x_i, outdegree 2 with an edge each for $x_i = 0$ and $x_i = 1$.

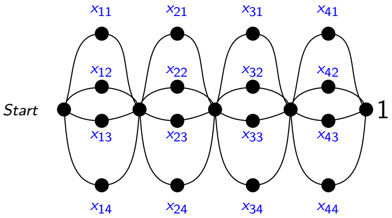


Non-det Branching Programs

Definition

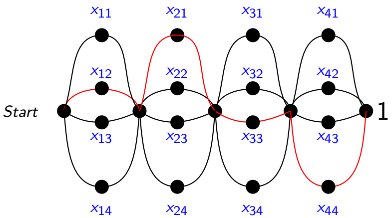
Non-deterministic Branching program (NBP)

- allow unlabelled guessing nodes and arbitrary out-degree.



The size of a NBP= number of labelled nodes.

NBP computing $f : \{0,1\}^n \rightarrow \{0,1\}$



$f(u) = 1 \iff$ $\exists$ a path from source to accept node that is consistent with input u.

Program Size and Space Complexity of computing f

- $BP(f_n) = \min_{B \in \text{ BP computing } f_n} \text{size}(B)$

- $S(f_n) = \min_{T \in \text{ non-uniform TMs computing } f_n} \text{space complexity}(T)$

- $\log(BP(f_n)) \approx S(f_n)$ [Cobham '66]

Big Picture

- It is easy to show functions with high $BP(f_n)$ exist.

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- Can we show that some function in P requires exponential size BP ?
  - amounts to showing $\mathrm{L} \subset \mathbf{P}$.

BPs and other Computation Models

Formulas		BranchingPrograms		Circuits
$L(f)$	$\geq$	$BP(f)$	$\geq$	$\frac{1}{3}C(f)$

BPs and other Computation Models

Formulas		*BranchingPrograms*		*Circuits*
$L(f)$	$\geq$	$BP(f)$	$\geq$	$\frac{1}{3}C(f)$
$\Omega\left(n^3\right)$		$\Omega\left(\frac{n^2}{\log^2 n}\right)$		$\Omega\left(n\right)$
Random Restrictions		Nechiporuk		Gate Elimination

Restricted Branching Programs

- Bounded Width: same as NC^1, Barrington's characterization.

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- Length Restricted: give Time-Space tradeoffs.

Time-Space tradeoffs

- $t \leq cn \implies s = 2^{\Omega(n)}$ Jukna'09

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- culmination results by Ajtai '99 and Beame,Jayram, Saks '01
- We look at:

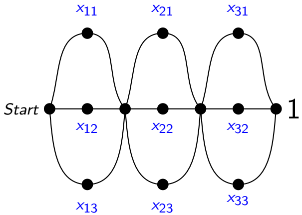
  time-space tradeoffs

  for

  iterated function composition.

Read Once

- *Syntactic* read once: Along any path from source to sink any variable appears atmost once.



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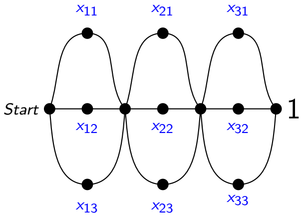

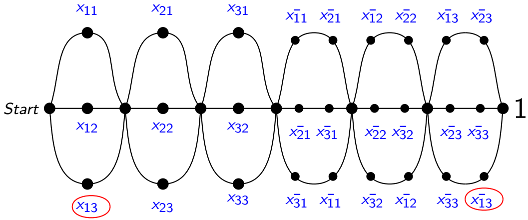

- *Semantic* read once: Along any *consistent* path from source to sink no variable is read more than once.

SYNTACTIC WEAKER THAN SEMANTIC

The *Exact Perfect matching* function(EPM_n): accept a matrix iff it is a permutation matrix.

Jukna and Razborov '98 showed

Theorem

Every syntactic read once NBP computing EPM_n must have size $2^{\Omega(n)}$.

$EPM_n \in$ SEMANTIC READ ONCE

THEOREM (JUKNA)

EPM_n can be solved by a semantic read once NBP of size $O(n^3)$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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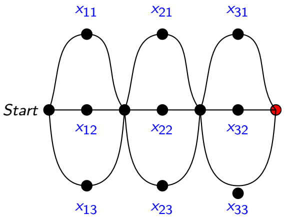

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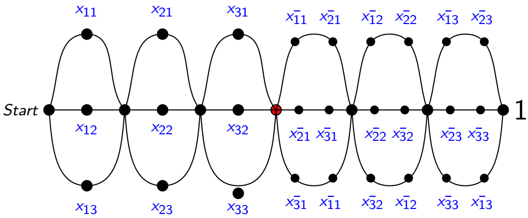

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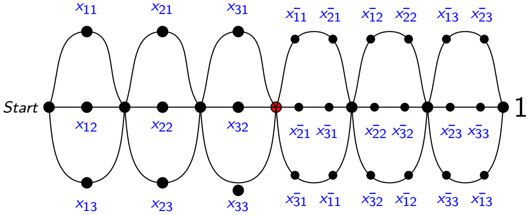

Sees only 1s

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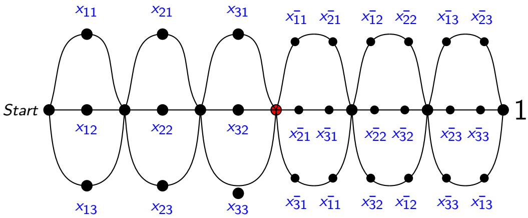

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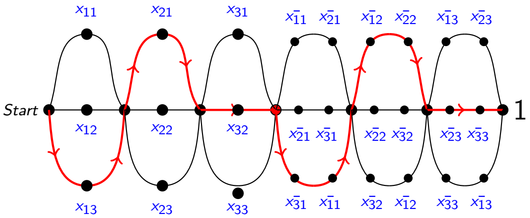

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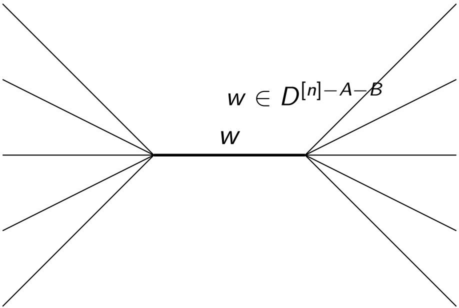
Embedded Rectangles
$C_{red} \times \{w\} \times C_{blue} \subseteq [D]^n$



$C_{red} \subseteq D^A$ $\quad$ $C_{blue} \subseteq D^B$

$A \subset [n]$ $\quad$ $B \subset [n]$

$$\left\{\begin{matrix} 012210101 \\ 012010201 \\ 101012202 \end{matrix}\right\} \times \{10210001\} \times \left\{\begin{matrix} 012001012 \\ 201101002 \\ 110020120 \\ 210200120 \end{matrix}\right\}$$

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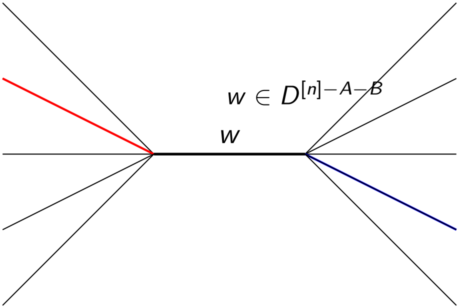

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$$\left\{\begin{matrix} 012210101 \\ \color{red}{012010201} \\ 101012202 \end{matrix}\right\} \times \{102101\} \times \left\{\begin{matrix} 012001012 \\ 201101002 \\ \color{blue}{110020120} \\ 210200120 \end{matrix}\right\}$$

KRW Conjecture

- an approach to separating NC^1 from NC^2.
- KRW conjecture that for every random f and $\forall\ g$, $D(fog) \geq \epsilon D(f) + D(g)$.
- KRW conjecture on formula size of a composed function fog.
  $L(fog) \approx L(f)L(g)$?

Understanding Composition

- How does space compose ?

Cook et.al 2012

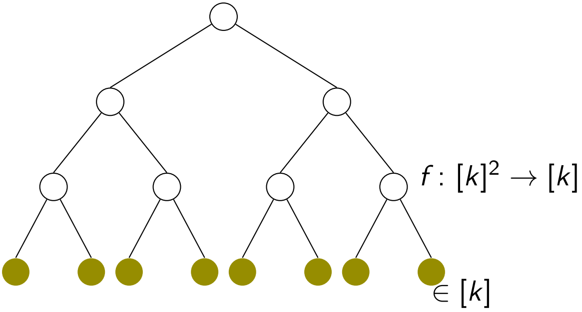
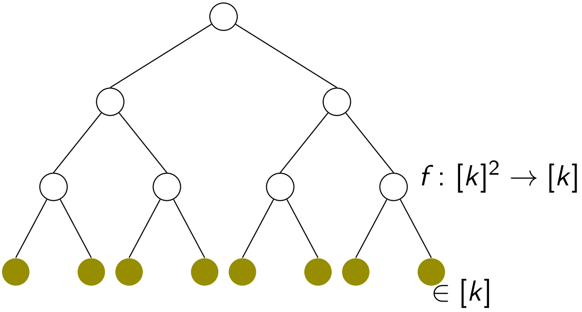


Figure: TEP_2^4 that is height 4, degree 2.

Cook et.al 2012



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Is $BP(TEP_2^h) = \Omega(k^h)$??

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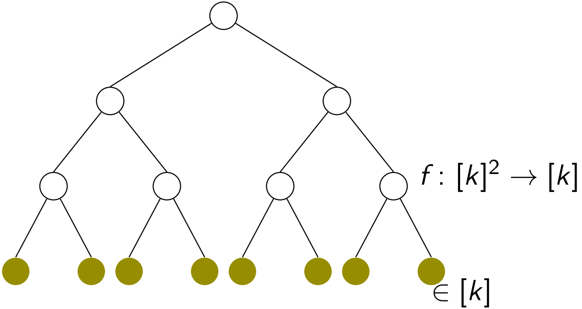
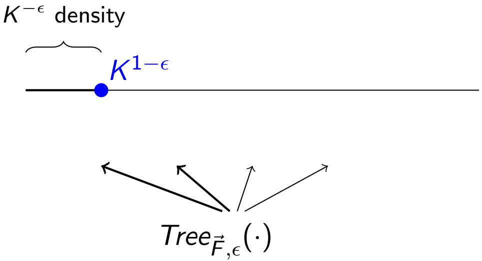


Figure: TEP_2^4 that is height 4, degree 2.

Is $BP(TEP_2^h) = \Omega(k^h)$?? $\implies$ $\mathsf{L} \subset \mathbf{P}$

Boolean TEP

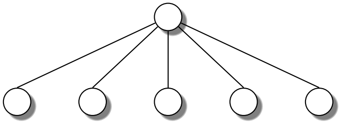


Lower Bound for composition

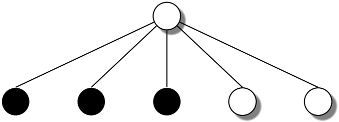
Theorem

For any h, and k sufficiently large, there exists ϵ and $\vec{F}$ such that any k-ary nondeterministic semantic read-once branching program for ternary $Tree_{\vec{F},\epsilon}$ requires size at least $\left(\frac{k}{\log k}\right)^h$.

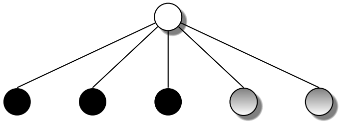
Black White pebbling Upperbound



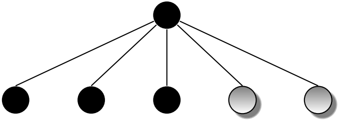
$\frac{1}{2}(d-1)h+1$ PEBBLES AT THIS MOMENT



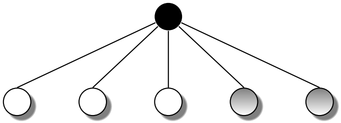
Guess the remaining siblings



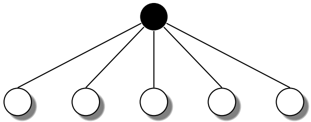
Infer the root



Unpebble blacks



Verify Guesses



DEFINITION

A Latin Cube is a function $f : [k]^3 \to [k]$ such that f is invertible in each of its coordinates.
Equivalently, every element of $[k]$ appears exactly once along every row,column and leg in the cube $[k]^3$.

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4-INVERTIBLE FUNCTION, $f: [k]^3 \to [k]$

Definition

Any element in $[k]$ appears at most 4 times along any row,column or leg in the cube $[k]^3$.

Proof Overview

- A small BP for a $Tree_{\vec{F},\epsilon}$ $\Longrightarrow$ $Tree_{\vec{F},\epsilon}$ accepts a large rectangle of inputs over its leaves

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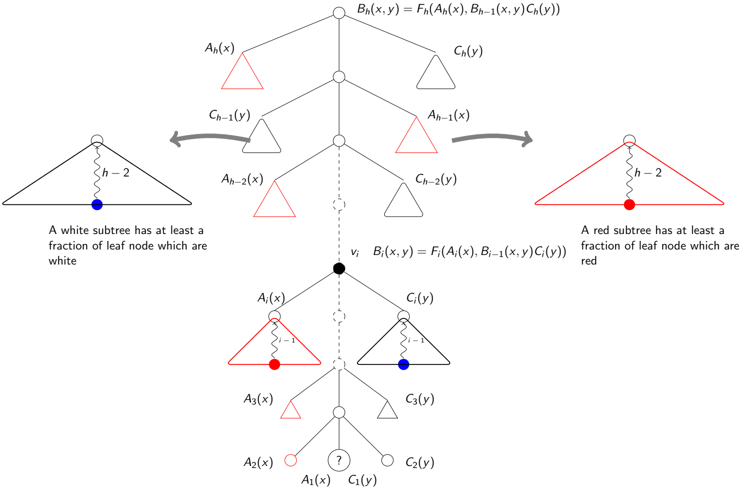
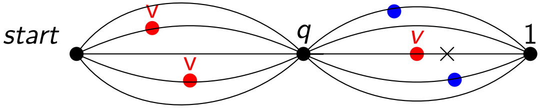
Proof Overview

- A small BP for a $Tree_{\vec{F},\epsilon}$ $\Longrightarrow$ $Tree_{\vec{F},\epsilon}$ accepts a large rectangle of inputs over its leaves
- A large rectangle over leaves $\Longrightarrow$ $\exists$ a special node v* in the tree whose F_{v*} can be described in few bits.
- Show that the distribution on $\vec{F}$ is rich or sufficiently random looking that one cannot save these bits.

FOR EVERY ACCEPTING INPUT ∃ A SPECIAL QUERY STATE:

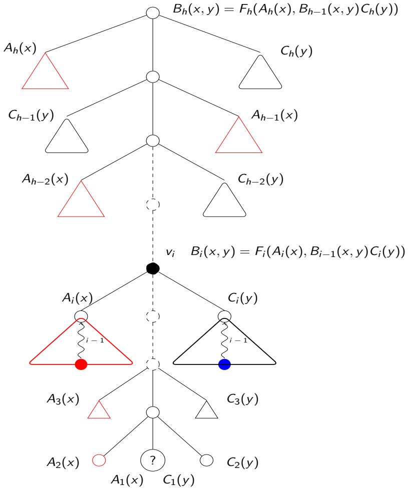


FEW SPECIAL STATES $\implies \exists$ A LARGE EMBEDDED RECTANGLE OVER LEAVES

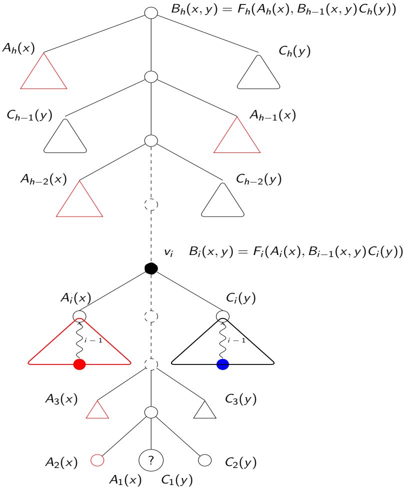


- Choose a popular labelled path down the tree.
- Choose a popular red variable for the first red-subtree. Prune the input set. Continue to choose h red variables one for each red-subtree. Similarly for each blue-subtree.
- Fix the remaining variables in $[n]$-*Red*-*Blue* to the most popular projection 'w'.

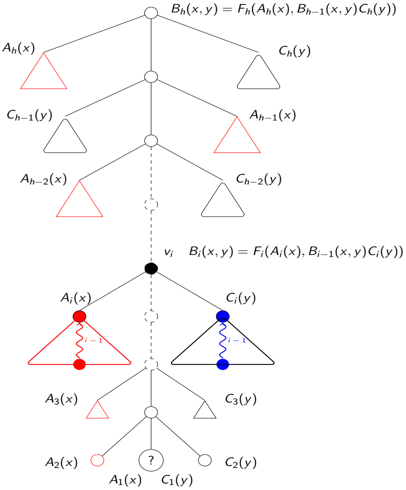
∃ A LARGE EMBEDDED RECTANGLE OVER THE LEAVES.



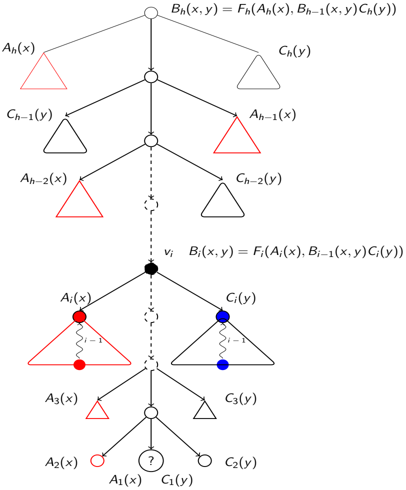
∃ a node $v*$ at which leaves in both red and blue trees take a lot of values.



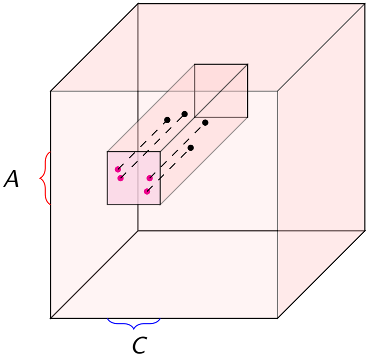
∃ A NODE $v*$ AT WHICH BOTH RED AND BLUE CHILD TAKE A LOT OF VALUES.



∃ A NODE $v*$ WITH LOW ENTROPY ON A TWO-WAY PRODUCT SET

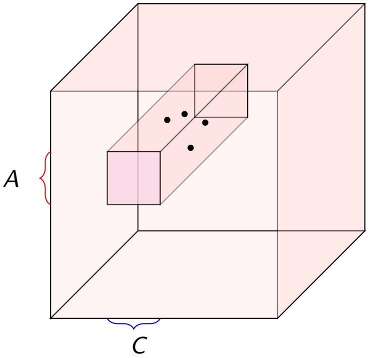


Two-way Product Set at $v*$, in $F_{v*}()$



$A, C \subset [k]$ $|A| = |C| = r << k$

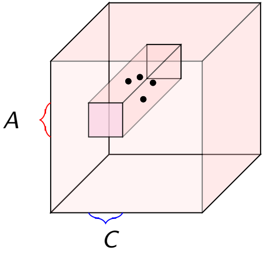
Two-way Product Set at $v*$, in $F_{v*}()$



$A, C \subset [k]$ $\quad\quad |A| = |C| = r << k$

$S_r = \{(x, Q(x,y), y) | x \in A, y \in C\}$ $\quad\quad |S| = r^2$

Enropy of spread on Two-way Product Set can't be low



$A, C \subset [k]$ $|A| = |C| = r << k$

$S_r = \{(x, B(x,y), y) | x \in A, y \in C\}$ $|S| = r^2$

$\forall$ Two-way Product Sets S_r and target set T_ϵ

$$\Pr_{f \sim U(All\ \ 4\text{-}invertible\ \ cubes)} [f(S_r) \subset T_\epsilon] \leq \frac{1}{k^{\epsilon r^2}}$$

Label

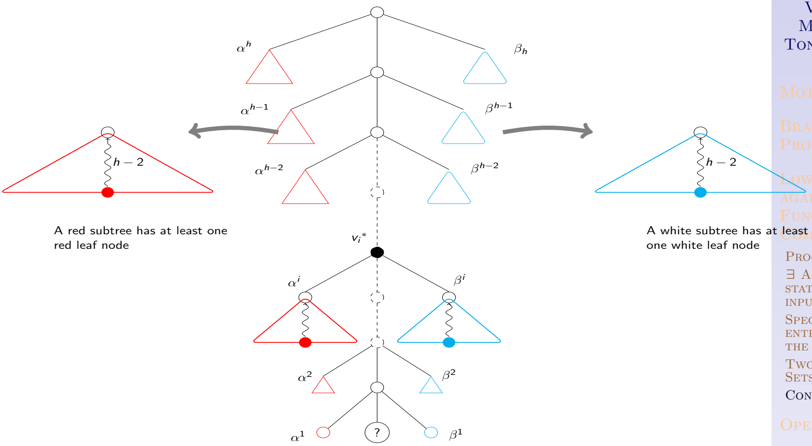
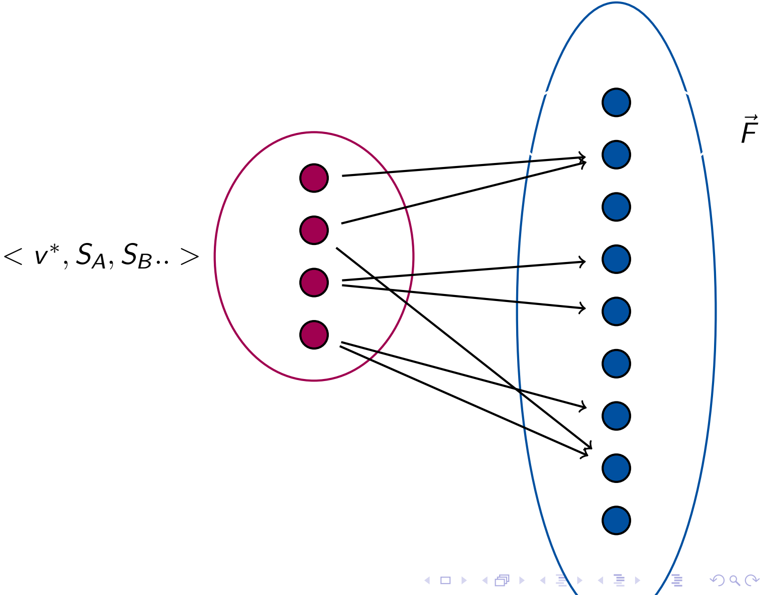


Figure: This figure depicts a *label* $L_{\vec{F}}$ associated with a problem instance $Tree_{\vec{F}}$

∃ many $\vec{F}$ that remain unaccounted without such a special label.



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Open Problems

- More general time space tradeoffs for composition.
- Exponential lower bound for boolean semantic NBPs for some problem in P.
- super-quadratic lower bound for BPs via understanding composition fog where g is element distinctness.

Thank You !