Pseudorandom generators from polarizing random walks

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Outline

Introduce Pseudorandom generators (PRGs)

New approach to construct PRGs

Open problems

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Introducing Pseudorandom generators(PRGs)

Goal: Construct random variable X.

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s *is called seed length*

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- Initiated by [Naor-Naor'90], found many applications

Fractional PRGs

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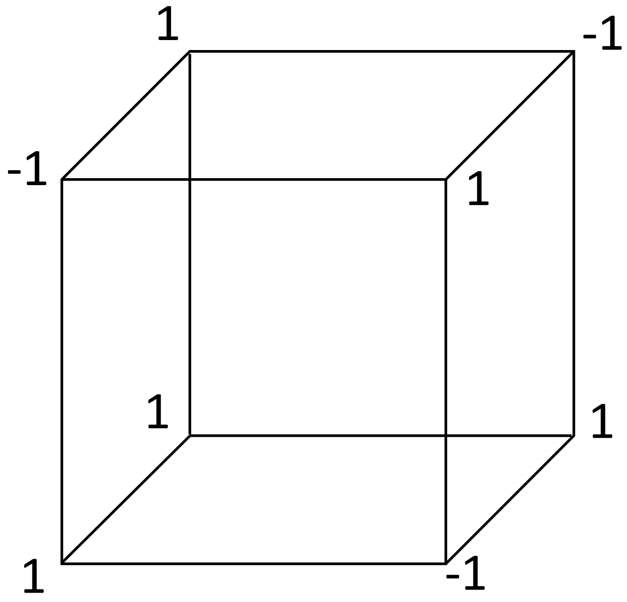

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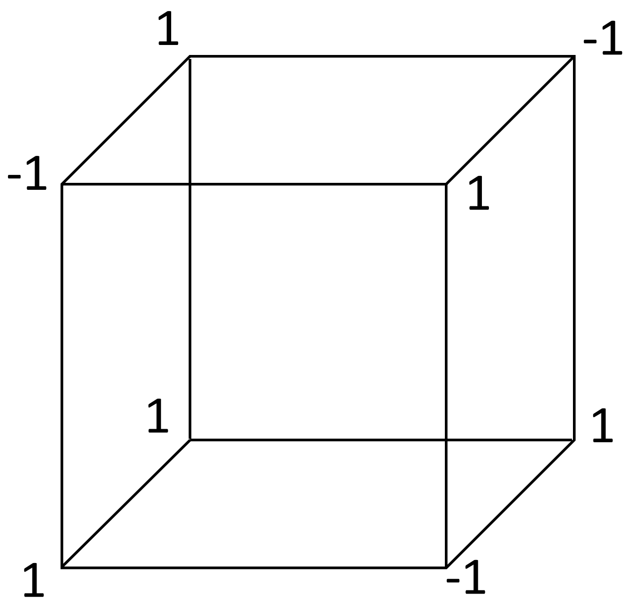

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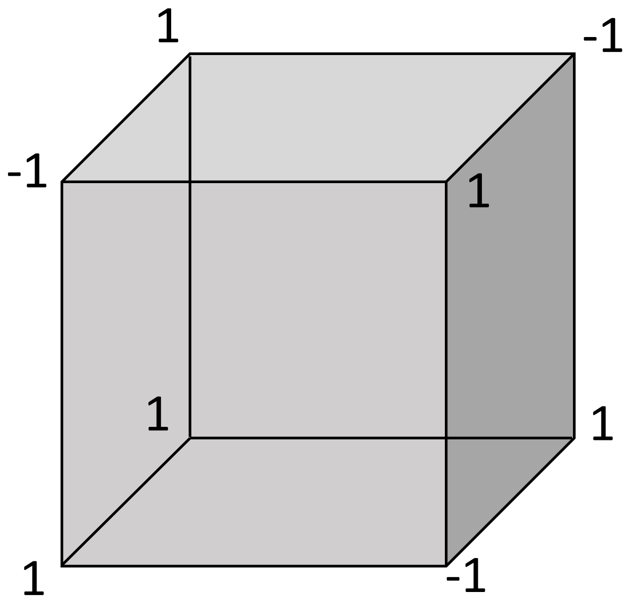
Only consider points in $[-1,1]^n$ so $f: [-1,1]^n \to [-1,1]$



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Equivalent definition of PRG:

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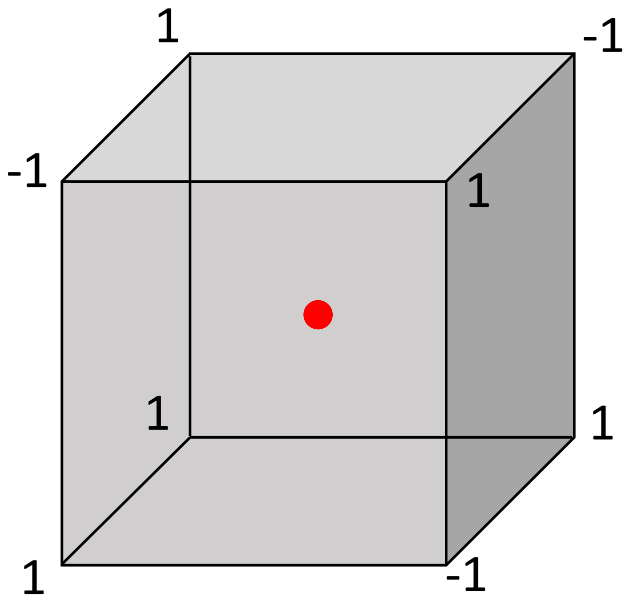

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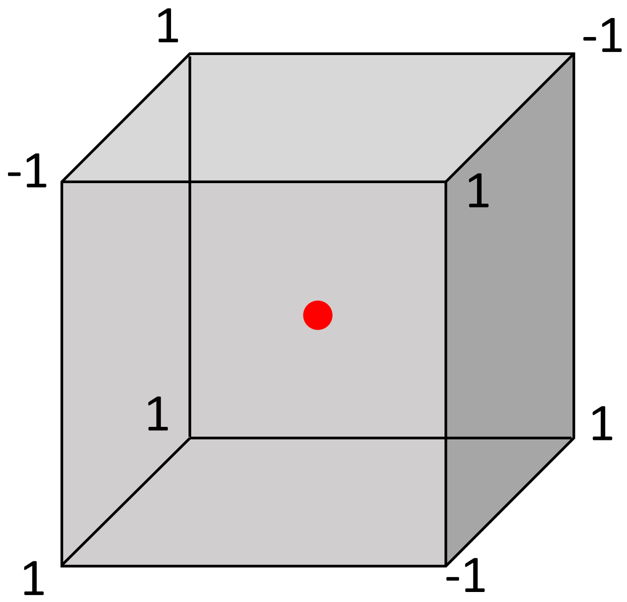

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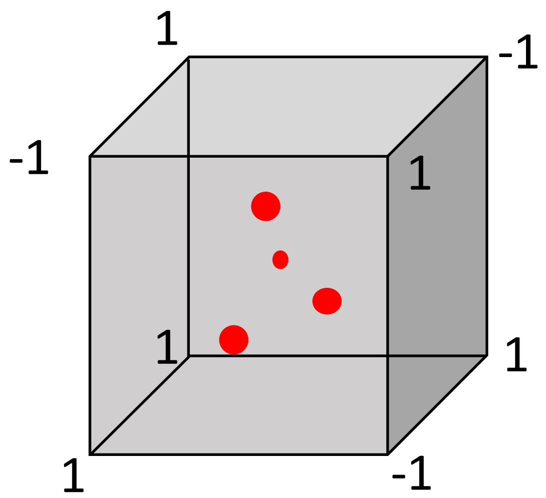

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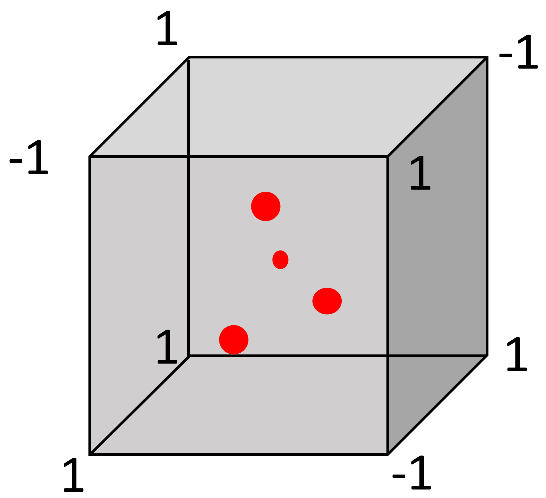

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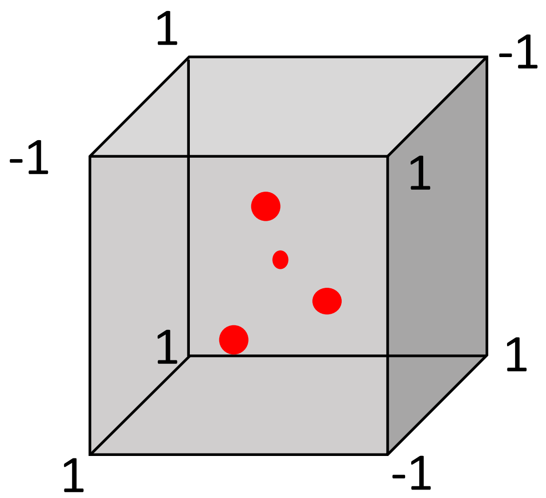

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Question. Are f-PRGs easier to construct than PRGs?

Can f-PRGs be used to construct PRGs?

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Recall: f-PRG is $X = (X_1, \cdots, X_n) \in [-1,1]^n$ where $|\mathbb{E} f(X) - f(0)| \leq \varepsilon$

Trivial solution: $X \equiv 0$

Need to enforce non-triviality: require $\mathbb{E}\,|X_i|^2 \geq p$ for all $i = 1, \ldots, n$

Constructing PRGs from f-PRGs

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- If X has seed length s then X' has seed length ts

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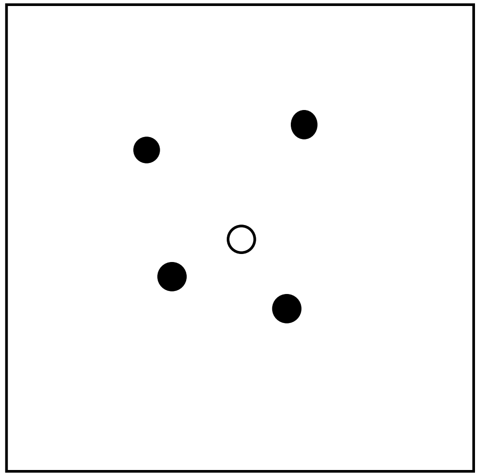
Goal: use the f-PRG to define a random walk



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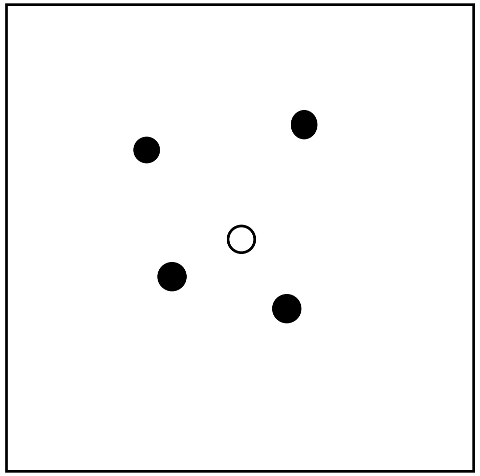

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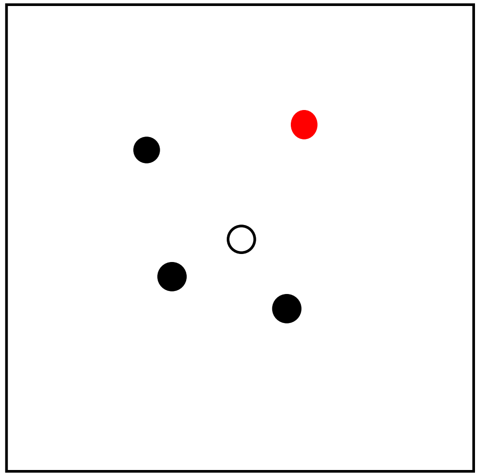

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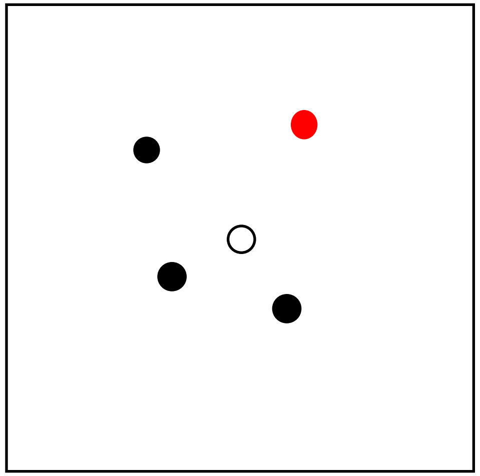

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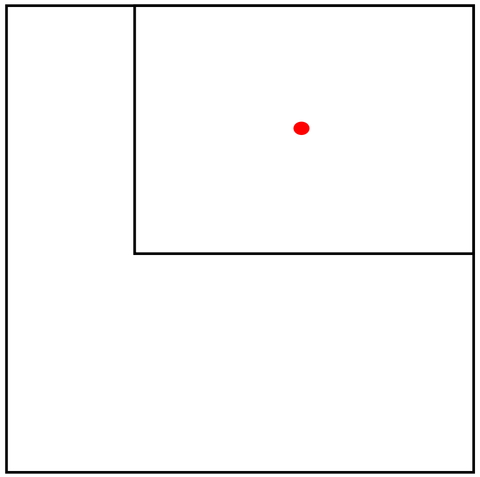

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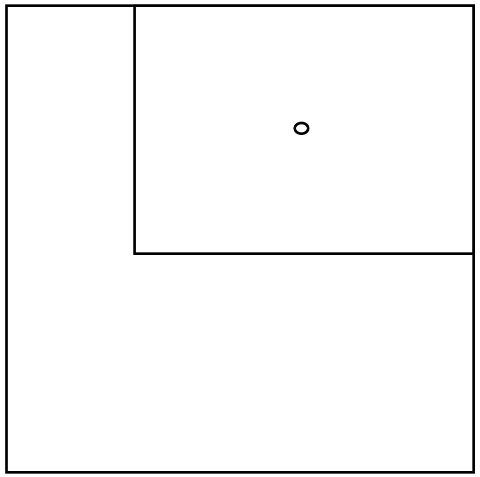

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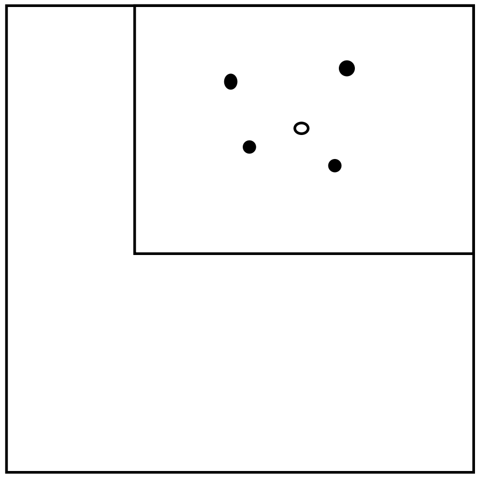

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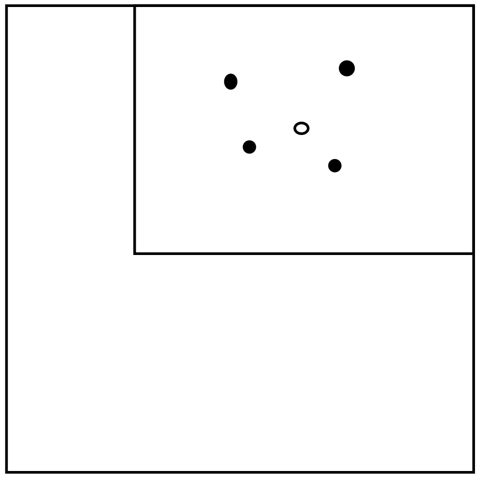

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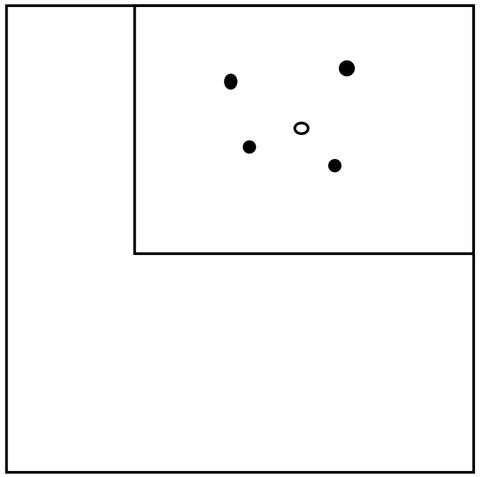

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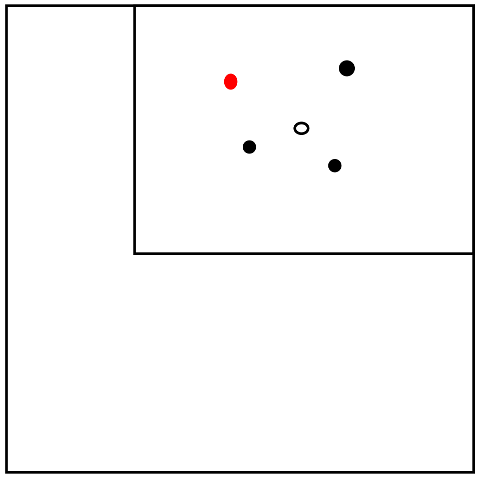

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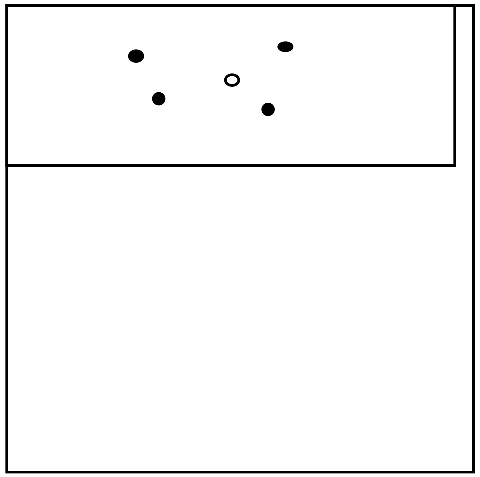

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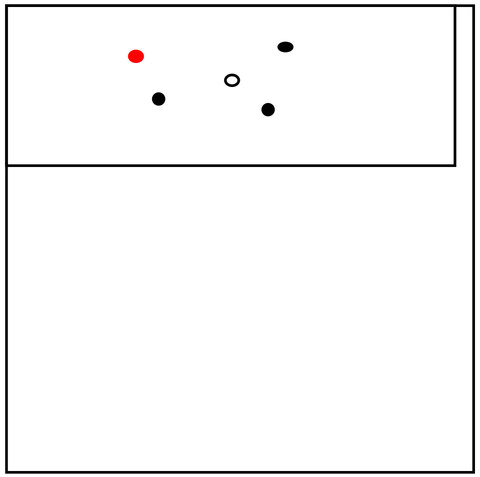

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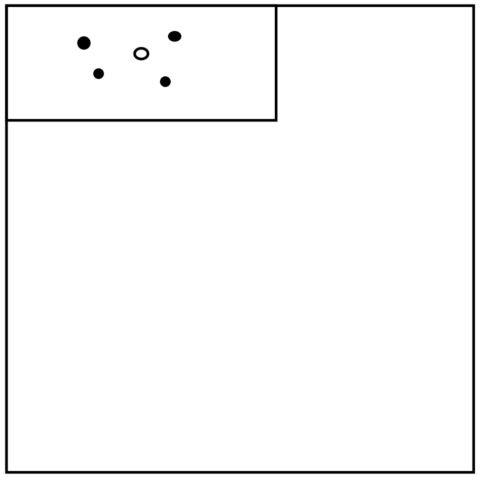

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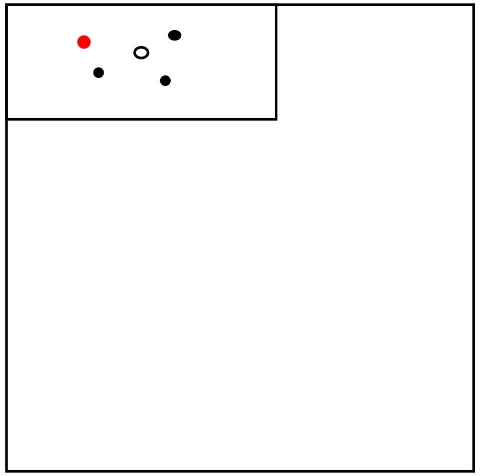

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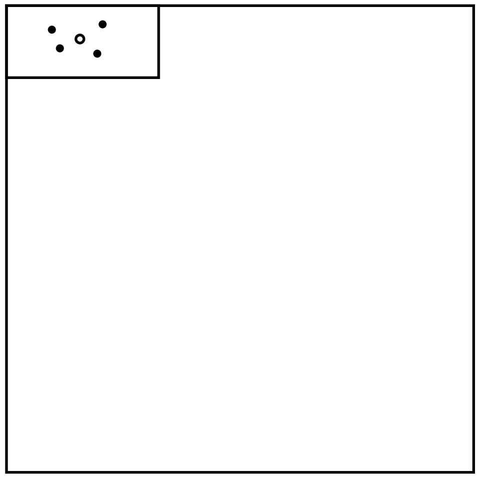

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Then after $O\left(\frac{1}{\mathbb{E}|X|^2} \log\left(\frac{1}{\varepsilon}\right)\right)$ steps, w.h.p $1 - |Y_t| \leq \varepsilon$

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Round to sign$\{Y_t\}$ once the random walk is close enough to the boundary

Construction of fractional PRGs

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$$f : \{-1,1\}^n \to \{-1,1\}$$

Fourier coefficients: $\hat{f}(S) = \mathbb{E}\, f(x) \prod_{i \in S} x_i \ , \quad S \subseteq [n]$

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f has bounded Fourier growth if

$$\sum_{S : |S| = k} |\hat{f}(S)| \leq c^k \qquad \forall k \geq 1$$

$\mathrm{c} = n$ is a trivial bound.

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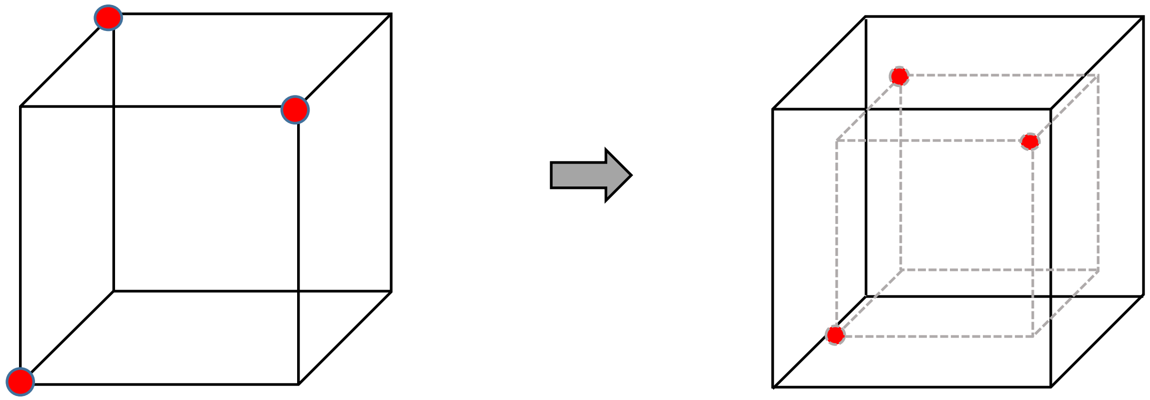

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Classes of functions:

Functions with sensitivity s:

$c = O(s)$ — Gopalan-Servedio-Wigderson'16

Permutation branching programs of width w:

$c = O(w^2)$ — Reingold-Steinke-Vadhan'13

Read once branching programs of width w:

$c = \log^w n$ — Chattopadhyay-Hatami-Reingold-Tal'18

Circuits of depth d:

$c = \log^d s$ — Tal'17

Questions

- One way to view our construction is as follows

X_1
$\vdots$
X_t

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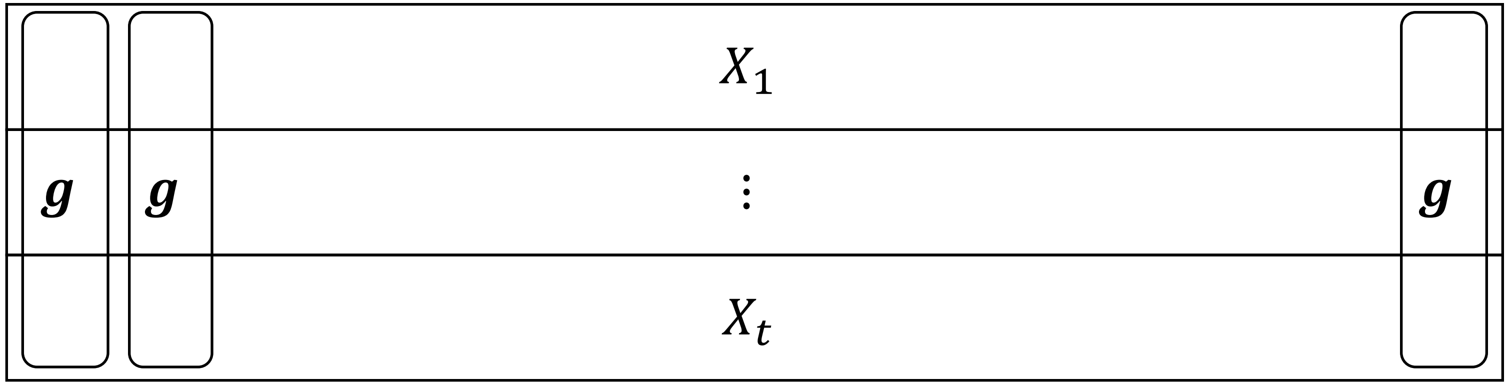

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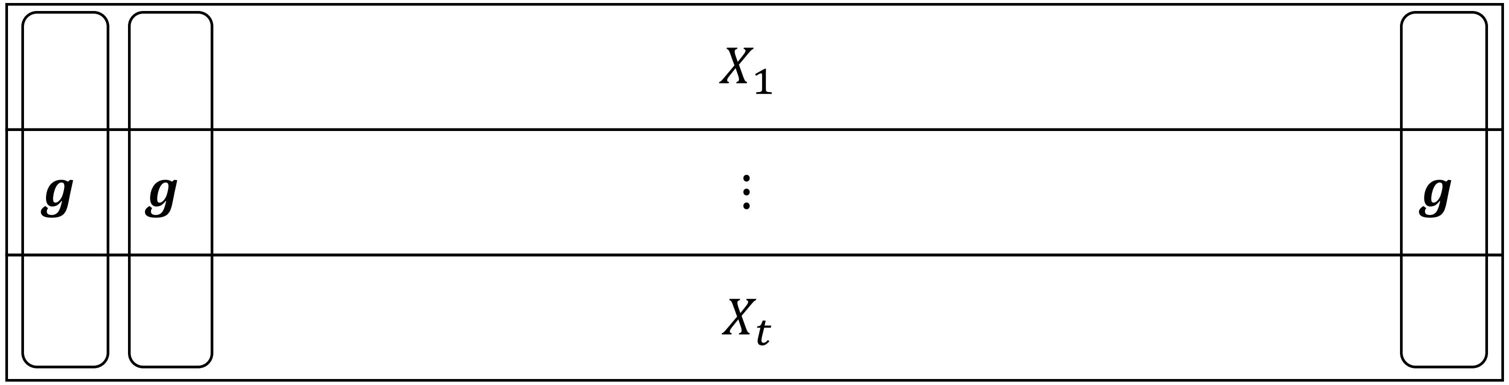

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$$G(X_1, \ldots, X_t) = \left(g(X_{1,1}, \ldots, X_{t,1}), \ldots, g(X_{1,n}, \ldots, X_{t,n})\right)$$

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- Can we construct other pseudorandom objects in this way?

Thank you!