UNBALANCING SETS AND AN ALMOST-QUADRATIC LOWER BOUND FOR SYNTACTICALLY MULTILINEAR CIRCUITS

Noga Alon
Princeton University

Mrinal Kumar
Harvard University

Ben Lee Volk
Tel Aviv University

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Galvin's problem:

What is the minimal $m = m(4k)$ such that there exists a family

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there is $j \in [m]$ such that $d_Y(S_j) = 0$?

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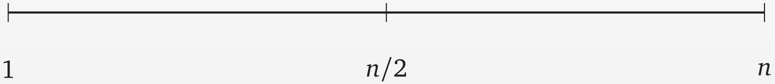

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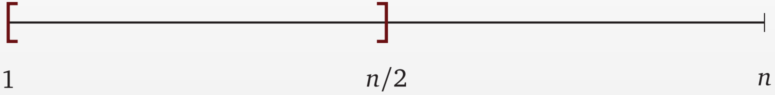

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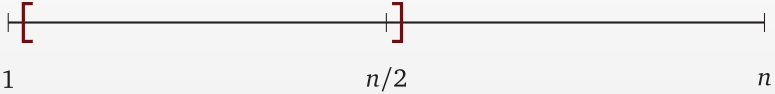

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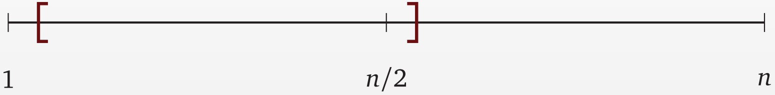

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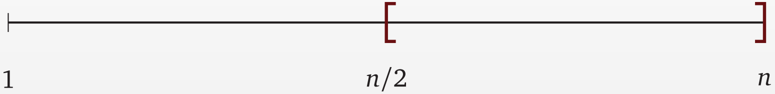

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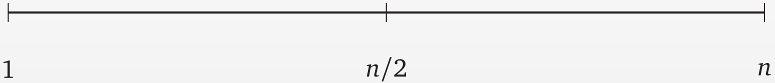

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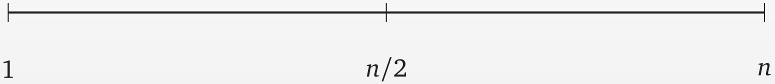

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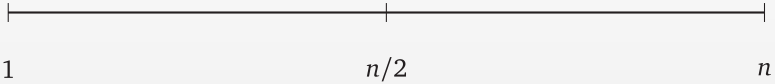

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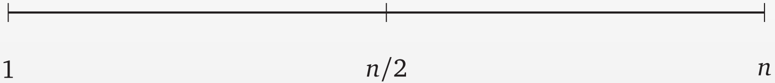

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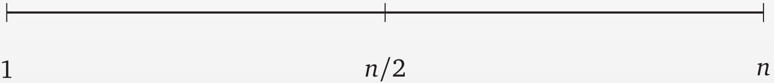

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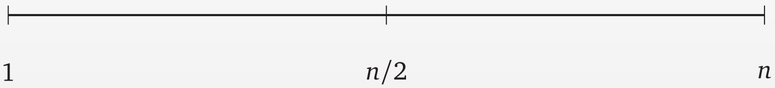

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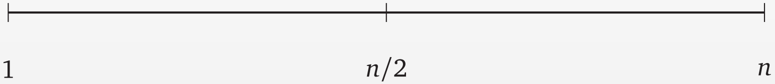

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(heads-up: circuit lower bound $\approx n \cdot m(n, \log n)$)

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Q is a non-zero polynomial which is zero on the middle layer of the boolean cube. Can we prove $\deg Q \ge$ anything?

Proof Sketch

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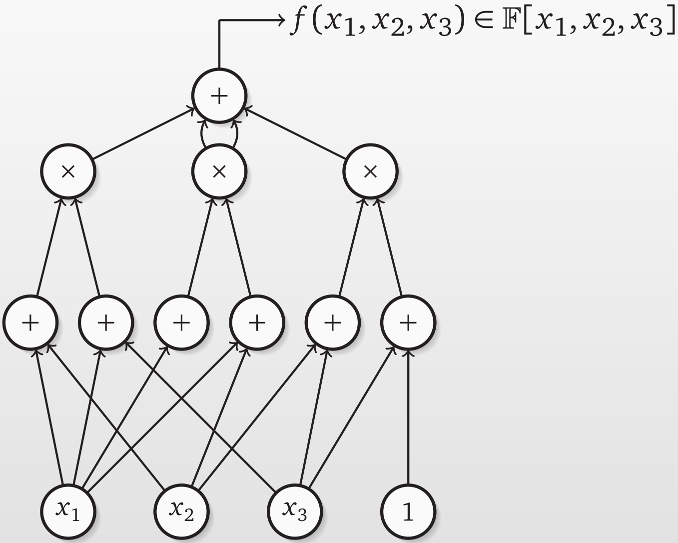
This is enough to prove a lower bound. More work required for general n and τ.

ALGEBRAIC CIRCUITS



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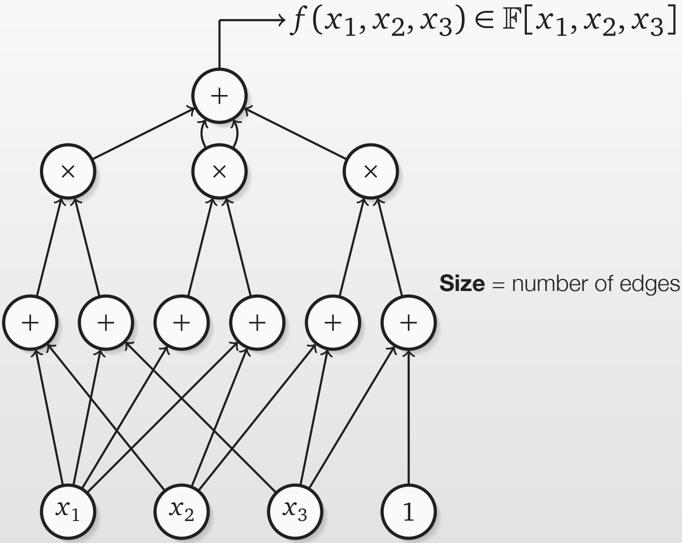

Size = number of edges

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[Raz-Yehudayoff]: $2^{\Omega(n^{1/d})}$ lower bound for depth-d formulas.

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This talk: $n \cdot m(n, \tau) = \tilde{\Omega}(n^2)$.

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We want to show this implies s must be large.

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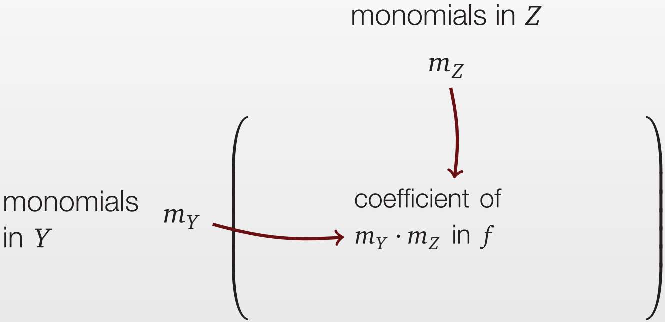

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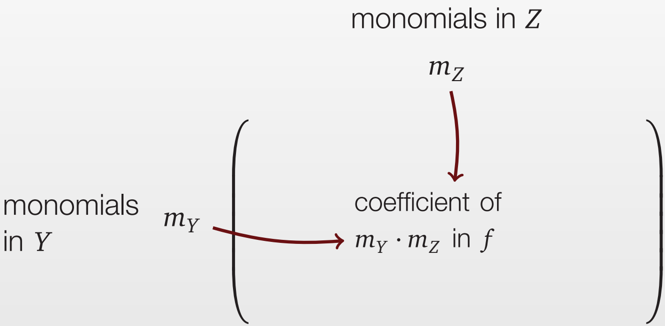

Let $\mathrm{rank}_{Y,Z}(f) = \mathrm{rank}\big(M_{Y,Z}(f)\big)$.

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So, if $f = \sum_{i=1}^{s} g_i h_i$, then

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If $s < m(n, \log n)$, we can find balanced Y such that for all i,

$$|Y \cap \mathsf{vars}(g_i)| \le |\mathsf{vars}(g_i)|/2 - \log n \quad \text{or} \quad \ge |\mathsf{vars}(g_i)|/2 + \log n,$$

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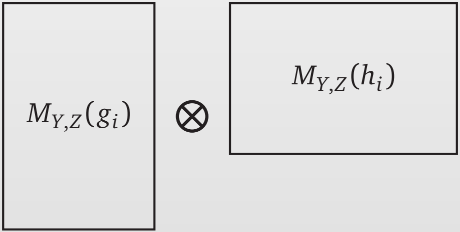
$\exists$ explicit f such that $\text{rank}_{Y,Z}(f) = 2^{n/2}$ for all balanced partitions.

If $s < m(n, \log n)$, we can find balanced Y such that for all i,

$$|Y \cap \text{vars}(g_i)| \leq |\text{vars}(g_i)|/2 - \log n \quad \text{or} \quad \geq |\text{vars}(g_i)|/2 + \log n,$$



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$\implies$ right hand side is $< 2^{n/2}$, unless $s \geq m(n, \log n)$. $\square$

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Thank You.