Derandomizing Isolation in Space-Bounded Settings

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Isolation

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- Computational Problem

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  (1) $|\Pi(f(x))| \leq 1$
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  (1) $|\Pi(f(x))| \leq 1$
  (2) $|\Pi(f(x))| = 0$ iff $|\Pi(x)| = 0$
  [(3) $\Pi(f(x)) \subseteq \Pi(x)$]

  With (3) f is called a *pruning*.

Motivation

- In algorithms:

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  Used to show problems become no easier when restricted to instances with unique solutions.

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- In complexity theory:

  Used to show problems become no easier when restricted to instances with unique solutions.

- All known generic isolations are randomized.

Space-Bounded Setting

- NL: languages accepted by non-deterministic logspace machines.
- UL: languages accepted by unambiguous logspace machines, i.e., non-deterministic machines with at most one accepting path on every input.

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- UL: languages accepted by unambiguous logspace machines, i.e., non-deterministic machines with at most one accepting path on every input.
- Open: $\mathrm{NL} \overset{?}{\subseteq} \mathrm{UL}$
  $\iff$ $\textsc{Reach} \in \mathrm{UL}$.
  $\iff$ $\textsc{Reach}$ has a logspace isolation.

Results

Theorem 1

$$\mathrm{NL} \subseteq \mathrm{UTISP}(\mathrm{poly}(n), O(\log^{3/2} n)).$$

Compare to

$\mathrm{NL} \subseteq \mathrm{DSPACE}(O(\log^2 n))$ [Savitch].

$\mathrm{Reach} \in \mathrm{DTISP}(\mathrm{poly}(n), n/2^{\sqrt{\log n}})$ [Barnes et al.].

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$\mathrm{NL} \subseteq \mathrm{L}/\mathrm{poly}$ if there is a logspace *pruning* for $\textsc{Reach}$.

Similar results for LogCFL.

Min-Isolation

Min-isolating weight assignment

For $G = (V, E)$, a weight assignment $w : V \to \mathbb{N}$ is *min-isolating* if for all $s, t \in V$, there is at most one min-weight s-t path.

The Isolation Lemma of Mulmuley, Vazirani and Vazirani implies For any graph G, a uniformly random choice of weights from $\{1, \ldots, n^3\}$ is min-isolating with high probability.

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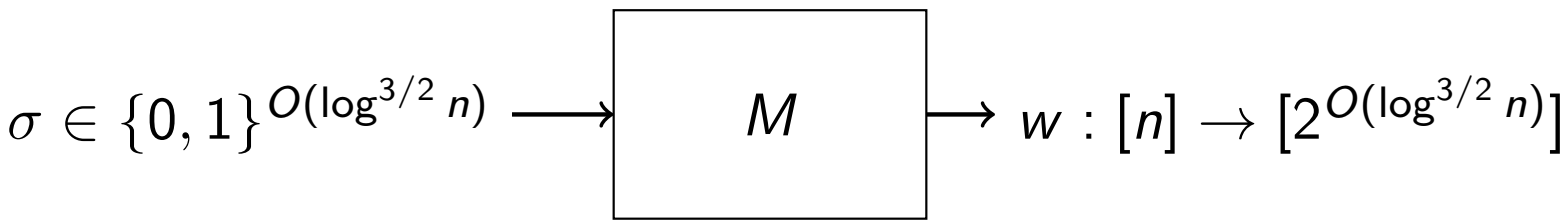
- Gal and Wigderson: $\mathrm{NL} \subseteq \mathcal{R} \cdot \text{promise-UL}$
- Reinhardt and Allender: $\mathrm{NL} \subseteq \mathcal{R} \cdot (\text{co-UL} \cap \mathrm{UL})$.

Derandomization

Key Lemma

There exists a logspace machine M

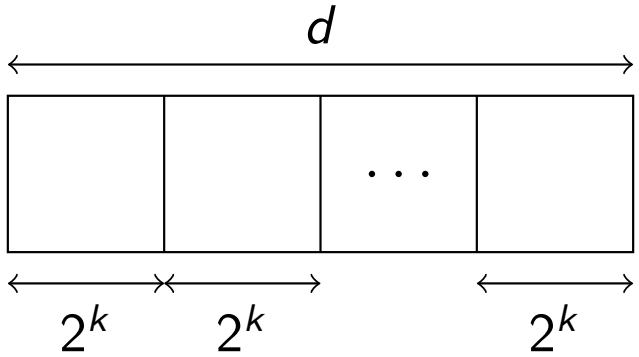


s.t. for all G on n vertices,

$$\Pr_{\sigma}[w \text{ is min-isolating for } G] \geq 1 - 1/n.$$

Proof of Key Lemma

Layered DAG:



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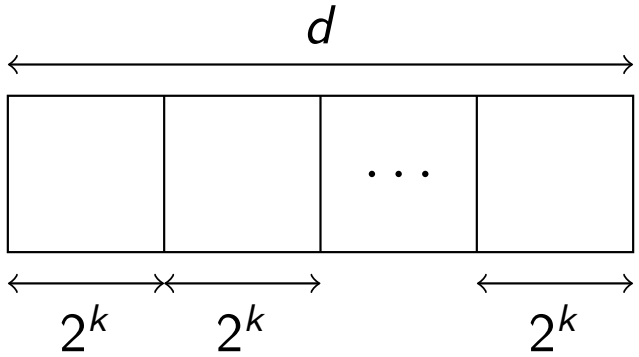

- Iteratively build $w_0, w_1, \ldots, w_{\log d}$.
- Invariant: w_k is
  - min-isolating for blocks of length 2^k.
  - non-zero only on internal vertices.

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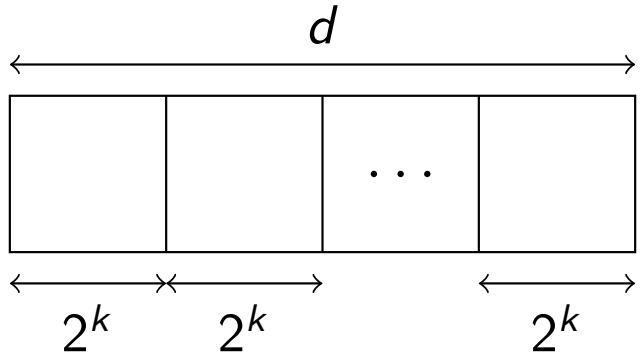

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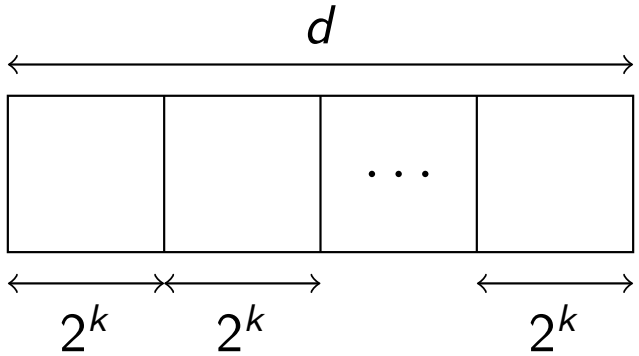

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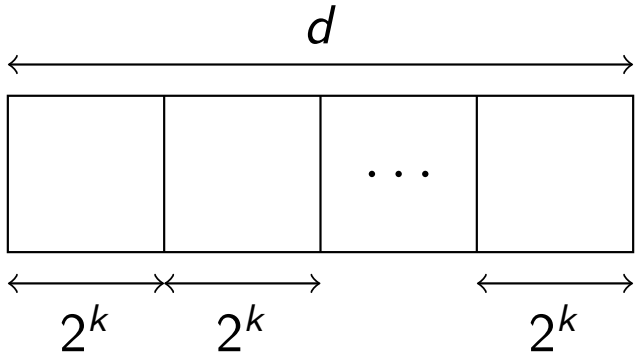

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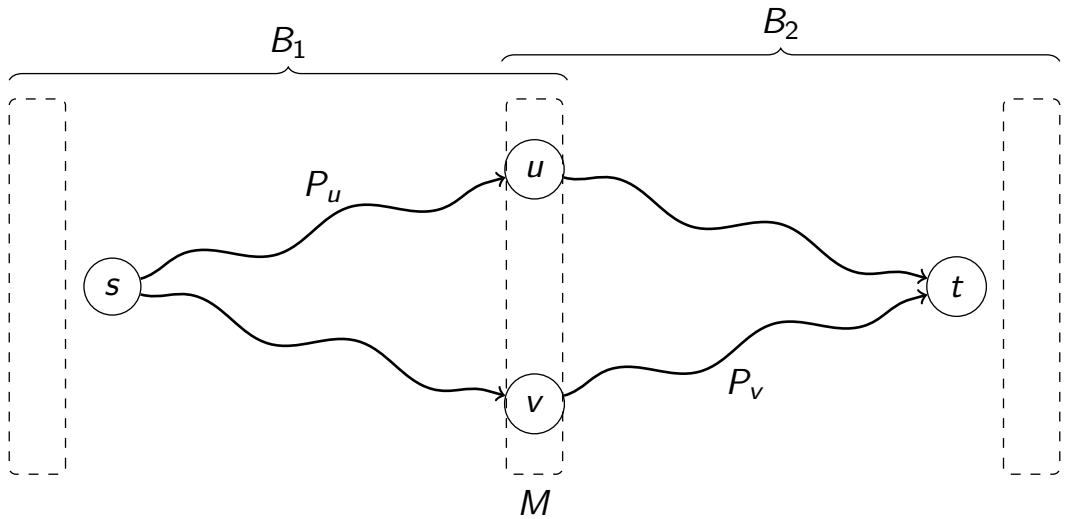
- Relevant Parameters:
  - Number of random bits R_k for w_k.
  - Maximum path weight W_k for w_k.
- Space used by unambiguous algorithm for $\textsc{Reach}$ is $O(R + \log W + \log n)$, where $R := R_{\log d}$ and $W := W_{\log d}$.

Iteration $k \to k+1$



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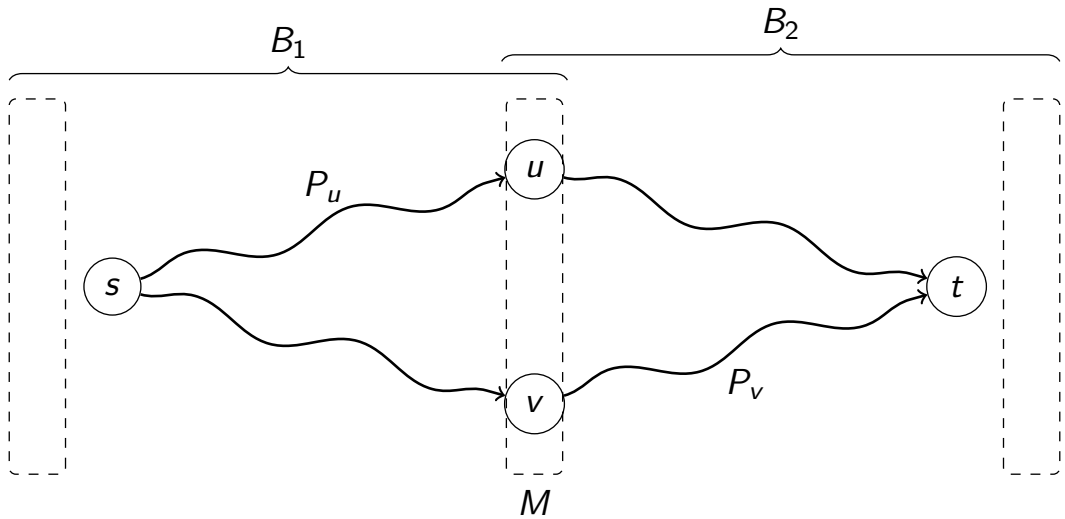

- w_{k+1}:
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  - Additionally assigns weights to L_{k+1},i.e., vertices in middle layers M of length 2^{k+1} blocks.

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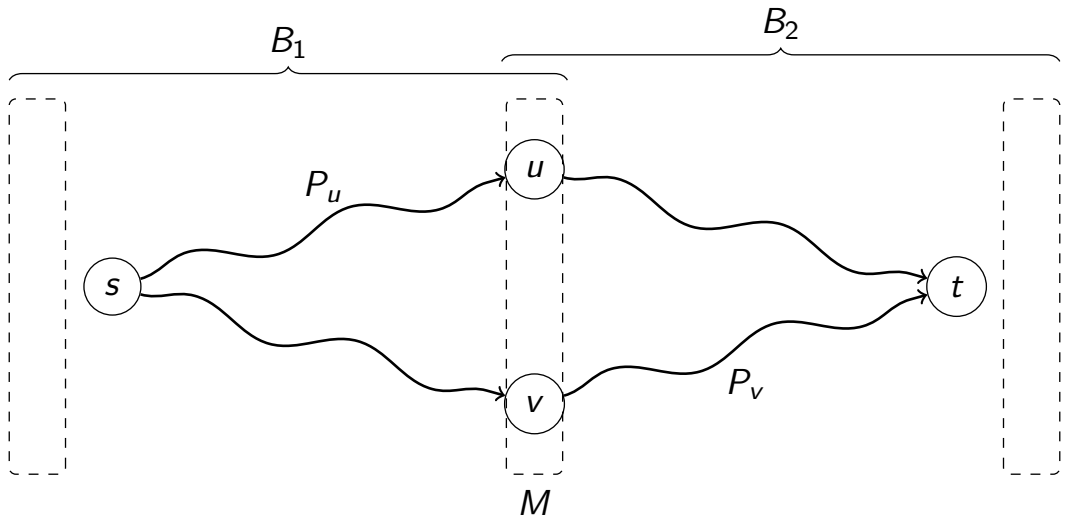

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- Disambiguation requirement:

$$w_k(P_u) + w_{k+1}(u) \neq w_k(P_v) + w_{k+1}(v)$$

$\forall s \in B_1, \forall t \in B_2, \forall u \neq v \in M, \forall$ blocks $B = B_1 \cup B_2$.

Shifting $k \to k+1$

$$w_k(P_u) + w_{k+1}(u) \neq w_k(P_v) + w_{k+1}(v)$$

Define

$$w_{k+1}(v) := \mathrm{index}(v) \cdot (W_k + 1) \text{ for } v \in L_{k+1}.$$

$$\begin{aligned} w_{k+1}(P_u) &= \mathrm{index}(u) \quad w_k(P_u) \\ &\neq \\ w_{k+1}(P_v) &= \mathrm{index}(v) \quad w_k(P_v) \end{aligned}$$

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- Parameters:

  $R_{k+1} = R_k \Rightarrow R_k = 0.$
  $W_{k+1} \leq n^2 \cdot (W_k + 1) \Rightarrow W_k = n^{O(k)}.$
  $R + \log W = O(\log^2 n).$

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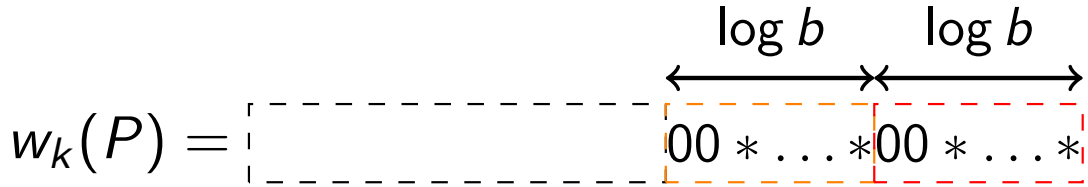
- Parameters:

  $R_{k+1} = R_k + O(\log n) \Rightarrow R_k = O(k \log n)$.
  $W_{k+1} = W_k + n^{O(1)} \Rightarrow W_k = k \cdot n^{O(1)}$.
  $R + \log W = O(\log^2 n)$.

Shifting and Hashing $k \to k'$



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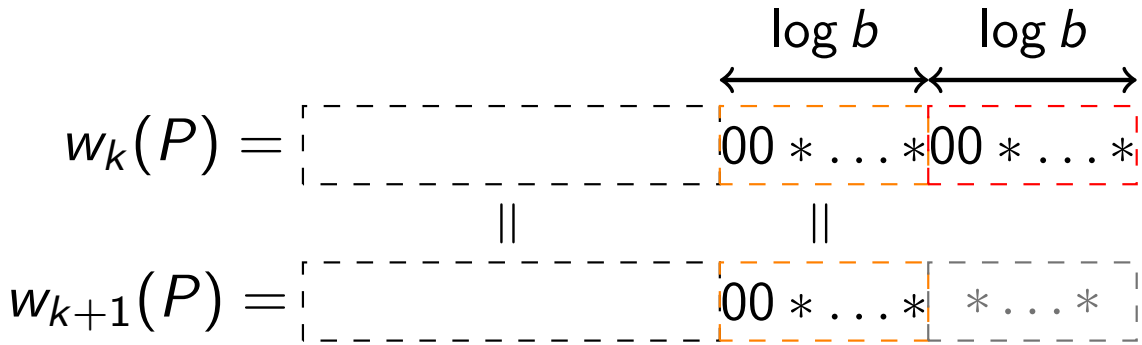

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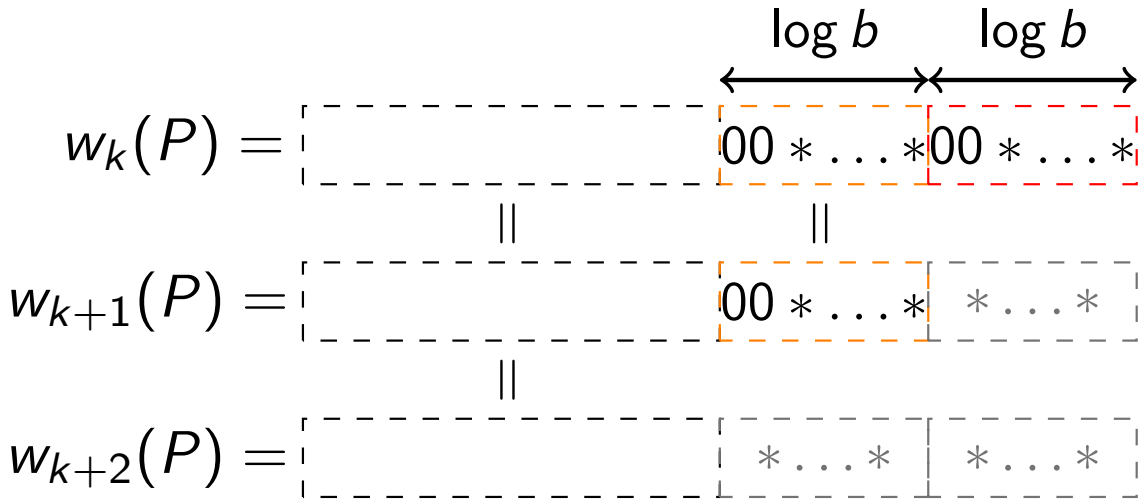

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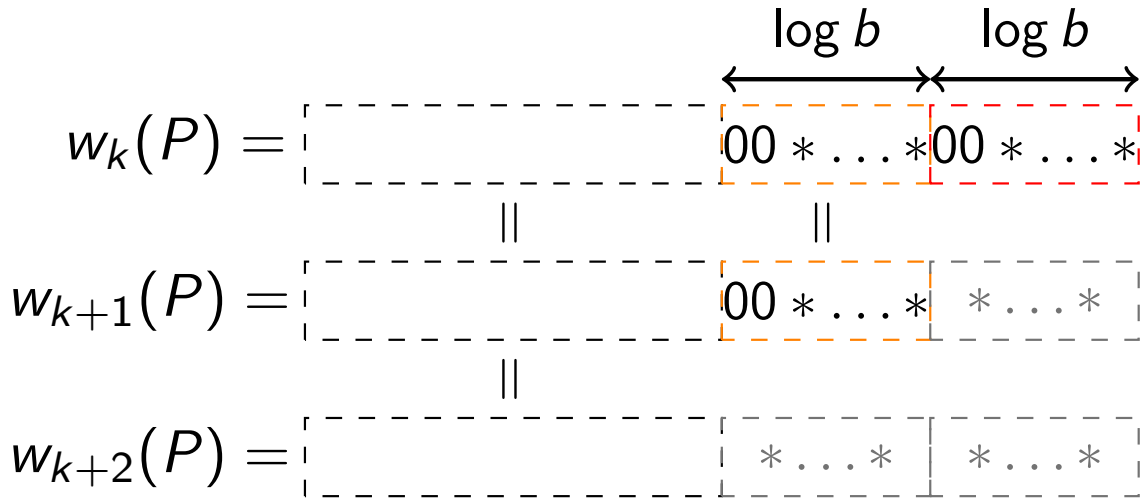

- For $i \in [k' - k]$,

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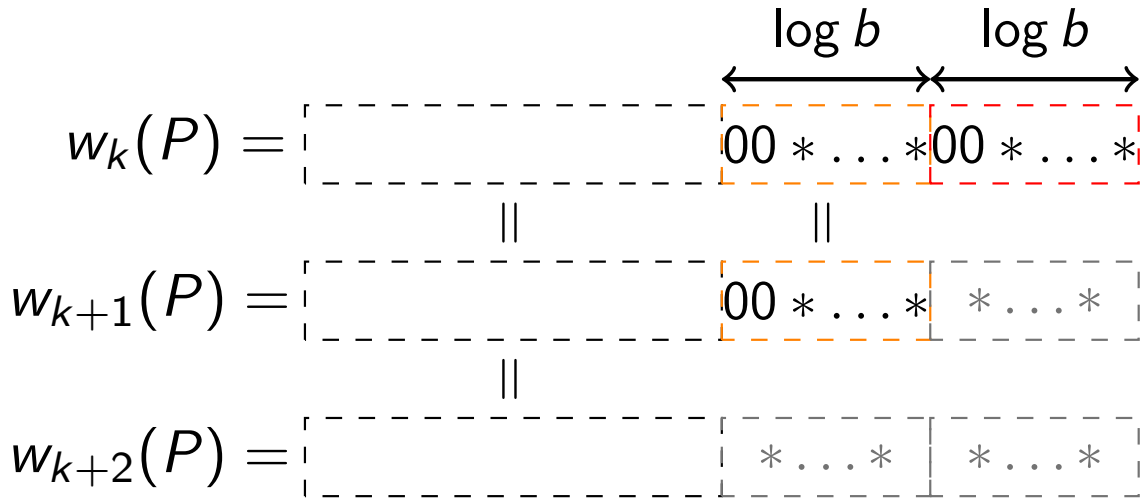

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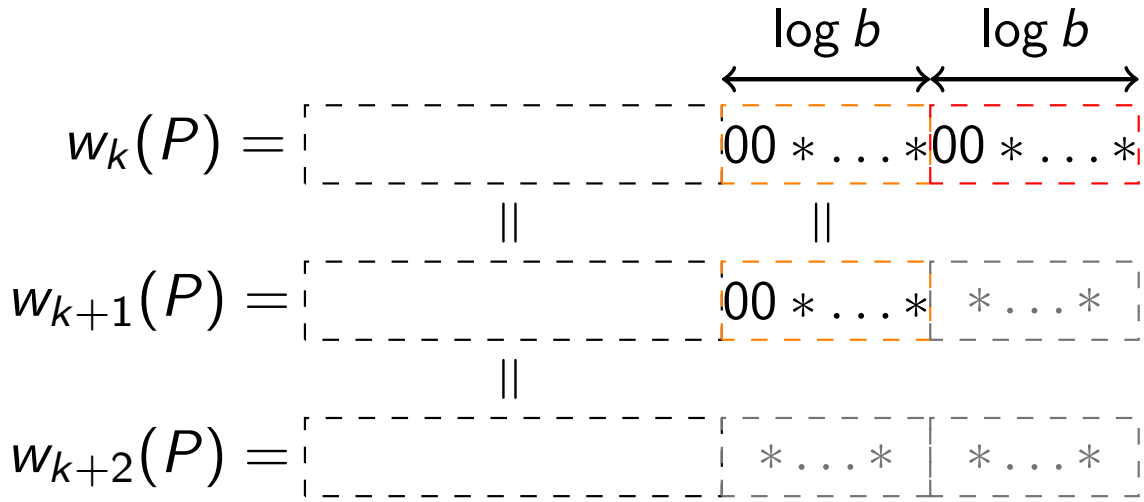

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- $R_{k'} = R_k + O(\log n)$.
  $W_{k'} = W_k + n^{O(k'-k)}$.
- Repeating with $\sqrt{\log d}$ hash functions and $k' - k = \sqrt{\log d}$ gives $R + \log W = O(\log n \sqrt{\log d}) = O(\log^{3/2} n)$.

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- Running through all possible choices of hash functions gives $\mathrm{NL} \subseteq \mathrm{USPACE}(O(\log^{3/2} n))$.
- Reduce running time to $\mathrm{poly}(n)$ by picking good hash functions one at a time.

Recap of main results

Theorem 1: $\mathrm{NL} \subseteq \mathrm{UTISP}(\mathrm{poly}(n), O(\log^{3/2} n))$.

Theorem 2: $\mathrm{NL} \subseteq \mathrm{L}/\mathrm{poly}$ if either of the following hold:

- there is a logspace *pruning* for $\textsc{Reach}$.
- there is a logspace algorithm that produces a polynomially bounded weight assignment w and a number that is equal to the weight of min-weight s-t path if there is a s-t path.

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- Open questions.

Thank You!

Questions?