Exponential lower bounds for hom. depth-5 circuits over finite fields

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Riga
Algebraic Circuits

\[ f(x_1, x_2, x_3) = 2x_2^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \]

\[ = 0 \text{ over } \mathbb{F}_2 \]
Algebraic Circuits

\[ f(x_1, x_2, x_3) \]

\[ = 2x_21 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \]

\[ \over \mathbb{F}_2 \]

\[ \text{Size} = \text{number of gates} \]
Algebraic Circuits

\[ f(x_1, x_2, x_3) \]

\[
\begin{align*}
&= 2x_2^1 + 2x_1x_2^1 + 2x_1x_3^1 + 2x_2x_3^1 \\
&= 0 \quad \text{over } F_2
\end{align*}
\]

Depth
Algebraic Circuits

\[ f(x_1, x_2, x_3) = 2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 0 \text{ over } \mathbb{F}_2 \]
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\( = 0 \text{ over } \mathbb{F}_2 \)
The Open Problem(s)

NP

\( \text{VP} \neq \text{VNP} \) is simpler to prove than \( \text{P} \neq \text{NP} \).

Ultimate goal: Find an explicit \( n \)-varied degree \( d \) polynomial that requires large arithmetic circuits to compute it.
The Open Problem(s)

\[ \text{VP} \subsetneq \text{VNP}, \text{P} \subsetneq \text{NP}. \]

Ultimate goal: Find an explicit \( n \)-variated degree \( d \) polynomial that requires large arithmetic circuits to compute it.
The Open Problem(s)

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Depth Reduction

Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by

algebraic circuits

of “small” size

Can be computed by

depth-4 circuits

of “not-too-large” size
Depth Reduction

Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by algebraic circuits of $\text{poly}(n, d)$ size

Can be computed by $\Sigma \Pi[\sqrt{d}] \Sigma \Pi[\sqrt{d}]$ circuits of $n^{O(\sqrt{d})}$ size
# Depth Reduction

**Theorem** ([Agrawal-Vinay + Koiran, Tavenas])

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(Or)

- Can be computed by $\Sigma \Pi [\sqrt{d}] \Sigma \Pi [\sqrt{d}]$ circuits
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A brief history of related results

**Goal:** To prove an $n^{\omega(\sqrt{d})}$ lower bound for $\sum \Pi[\sqrt{d}] \sum \Pi[\sqrt{d}]$ circuits.
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A \( 2^{\Omega(d)} \) lower bound for \( \sum \Pi[d] \sum \) circuits.
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Theorem ([Kayal-Limaye-Saha-Srinivasan])
A $n^{\Omega(\sqrt{d})}$ lower bound for homogeneous depth-4 circuits.
Our results

**Theorem**
An explicit polynomial $f(x_1, \ldots, x_n)$ of degree $d$ with 0/1 coefficients such that, for any fixed finite field $\mathbb{F}_q$, any homogeneous $\Sigma \Pi \Sigma \Pi \Sigma$ circuit computing $f$ must have size $2^{\Omega_q(\sqrt{d})}$. 

Ingredients for the proof:
- [Kayal-Limaye-Saha-Srinivasan]
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... ought to have been easier than this
How are such bounds proved?

Natural proof strategies

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

1. If $f$ is computable by “small” circuits, then $\Gamma(f)$ is “small”.

2. For the desired polynomial $f$ we wish to show a lower bound, then $\Gamma(f)$ is “large”.
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Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that: Typically $\Gamma(f)$ is the rank of some associated linear space.

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**Key observation:** There are just $\binom{d}{k}$ linearly independent $k$-th order partial derivatives of $\ell_1 \cdots \ell_d$. 
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$$\partial^k (\ell_1 \cdots \ell_d) \subseteq \text{span} \left\{ \prod_{i \in S} \ell_i : S \subseteq [d], |S| = d - k \right\}$$
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For $f = \text{Det}_d$, the symbolic determinant of a $d \times d$ matrix, we have $\binom{d}{k}^2$ linearly independent $(d - k) \times (d - k)$ minors.
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Therefore, if \( \text{Det}_d = \sum_{i=1}^s \ell_{i1} \cdots \ell_{id} \), then \( s \geq \binom{d}{d/2} \). \( \square \)
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\partial_x (Q_1 \cdots Q_r) = \partial_x (Q_1) \cdot Q_2 \cdots Q_r + \cdots + Q_1 \cdots Q_{r-1} \cdot \partial_x (Q_r)
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$$\partial^k (Q_1 \cdots Q_r) = \text{span} \left\{ x^{k \sqrt{d}} \cdot \prod_{i \in S} Q_i : S \subset [r], |S| = r - k \right\}$$
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![Diagram of a multivariate polynomial with variables $x_1, \ldots, x_n$ and partial derivatives $\partial_{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} f$](image)
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  ![Diagram showing low degree monomials vs. high degree monomials]

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Dimension of shifted partials of $f$. 

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Dimension of shifted partials of a random restriction of $f$. 
Examples...

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  hom. $\Sigma\Pi\Sigma\Pi$ circuits: terms like $Q_1 \cdots Q_b$ with total degree $d$.

- **Idea 1 - Random Restrictions**: Randomly set a small number of variables to zero
- **Idea 2 - Multilinear Projection**: Discard all non-multilinear monomials

$$
\Gamma(f) = \dim(\text{mult} \circ x^{\ell} \partial^{k}(\rho(f)))
$$

Dimension of projected shifted partials of a random restriction of $f$. 
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Multilinear mons of degree $\ell + d - k$
Handling depth-5 circuits

We already have a complexity measure PSPD for hom. depth-4 circuits. How large is PSPD for a generic depth-5 circuit? Small (a lower bound against depth-5 circuits). Large (separation between depth-5 and depth-4 circuits).... still don't know
Handling depth-5 circuits

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Evaluating the complexity measure

\[ \Gamma_k(f) = \dim \left\{ \partial^k (f) \right\} \]
Evaluating the complexity measure

\[ \partial = k \]

Monomials of degree \( d - k \)

\[ \partial_{x^\alpha} \]

coeff. of \( m \)
in \( \partial_{x^\alpha}(f) \)
Evaluating the complexity measure

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Coeff. of \( m \) in \( \partial_{x^\alpha}(f) \)

Points in \( \mathbb{F}_q^n \)

\[ \partial = k \]

eval. of \( \partial_{x^\alpha}(\rho(f)) \) at \( \bar{a} \)
Evaluating the complexity measure

Monomials of degree $d - k$

Eval. of $\partial_x^\alpha (\rho(f))$ at $\bar{a}$

Small rank
\[ f = \ell_{11} \cdots \ell_{1d_1} + \cdots + \ell_{s1} \cdots \ell_{sd_s} \]
Grigoriev-Karpinski

\[ f = \ell_1 \cdots \ell_{d_1} + \cdots + \ell_s \cdots \ell_{d_s} \]

Low degree terms.

High degree terms.
Grigoriev-Karpinski

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- **Low degree terms.**
- **High degree terms:** high rank terms
- **High degree terms:** low rank terms

Eg. \( \ell_1^{d/3} \ell_2^{d/3} (\ell_1 + 3\ell_2)^{d/3} \)
Grigoriev-Karpinski

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Low degree terms. \[\text{High degree high rank terms} \]

High degree low rank terms

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- Low degree terms.
- High degree high rank terms
- High degree low rank terms

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[GW-95]
Grigoriev-Karpinski

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Low degree terms.

\[ \sqrt{[NW-95]} \]

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High degree low rank terms

Observation

If \( \dim \{ \ell_1, \cdots, \ell_r \} \) is large, then almost all evaluations of it on \( \mathbb{F}_q^m \) are zero.
Grigoriev-Karpinski

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\[ \partial = k \]

Mons. of degree \( d - k \)

\[ \partial_x^\alpha \]

coeff. of \( m \) in \( \partial_x^\alpha(f) \)
\[ \partial = k \]

\[ \bar{a} \]

\[ \partial_{\alpha} \]

\[ \text{eval. of } \partial_{\alpha}(f) \text{ at } \bar{a} \]

\[ \mathbb{F}^n_q \]
\[ \partial = k \]

\[ \partial_{x^\alpha} \]

Eval. of \( \partial_{x^\alpha}(f) \) at \( \tilde{a} \)

\[ \mathbb{F}^n_q \]
Lemma

If $f$ is computable by a small $\Sigma \Pi \Sigma$ circuit over $\mathbb{F}_q$, then there the above matrix has small rank when a certain small set of columns are removed.
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Lemma
For $\text{Det}_n$ or $\text{Perm}_n$ the above matrix remains full rank, as long as we removed only few columns.
Lifting to depth five

ΣΠΣΠΣ

Types of products of linear polynomials:

- Low degree products.
- High degree products.
Lifting to depth five

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Types of products of linear polynomials:

Low degree products.  

High degree products.

[GKKS]
Lifting to depth five

Types of products of linear polynomials:

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Lifting to depth five

$\Sigma \Pi \Sigma \Pi \Sigma$

Types of products of linear polynomials:

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If \( \text{dim} \{ l_1, \cdots, l_r \} \) is large, then almost all evaluations of it on \( \mathbb{F}_q^m \) are zero.
Lifting to depth five

\[\Sigma \Pi \Sigma \Pi \Sigma\]

Types of products of linear polynomials:

- Low degree products.
  - \([\text{GKKS}]\]
  - Eg. \(\ell_1 \cdots \ell_d\)
  - \(\sqrt{[\ell_1] \cdots [\ell_d]}\)

- High degree, large rank products.
  - Eg. \(\ell_1 \cdots \ell_d\)

- High degree, small rank products.
  - Eg. \(\ell_1^{d/2} \ell_2^{d/2}\)

Observation

If \(\dim \{\ell_1, \cdots, \ell_r\}\) is large, then almost all evaluations of it on \(\mathbb{F}_q^n\) are zero.
We know this rank is large:

\[ x = \ell \partial = k \]

\[ m \]

\[ x^\beta \partial_{x^\alpha} \]

Coeff. of \( m \) in \( x^\beta \partial_{x^\alpha}(f) \)

Mons of degree \( \ell + d - k \)
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Mons of degree \( \ell + d - k \)

Need to show this rank is large:

\[ x^\ell \partial = k \]

\[ \{0, 1\}^n \]

coeff. of \( m \) in \( x^\beta \partial_{x^\alpha}(f) \)

eval. of \( x^\beta \partial_{x^\alpha}(f) \) at \( a \)
Switching to the evaluation perspective

\[ x^\beta \partial_{x^\alpha} = \begin{cases} 
  x^\ell \partial^k 
  & \text{eval. of } x^\beta \partial_{x^\alpha}(f) \text{ at } a \\
  \mathbb{F}^n_q & 
\end{cases} \]

Mons of degree \( \ell + d - k \)
Switching to the evaluation perspective

Monsof degree $\ell + d - k$

Largerank $\therefore [KLSS, KS]$

$x^\beta \partial_{x^\alpha}$

$Vandermonde$

$= x = \ell \partial = k$

$\mathbb{F}^n_q$

Eval. of $x^\beta \partial_{x^\alpha}(f)$ at $a$
Switching to the evaluation perspective

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Switching to the evaluation perspective

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Mons of degree \( \ell + d - k \)

\[ = \quad x^{=\ell} \partial^{=k} \]

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Large rank \( \vdash [KLSS, KS] \)

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Issues to be resolved

Issue 1: If $\text{Fatmatrix}$ and $\text{Tallmatrix}$ could both be zero, even if both are full rank.

Fix: Make the matrix slimmer by only considering evaluations on $f_0, 1, g_{\text{full}}$.

Issue 2: But then $(x_1 + 1, x_n + 1)$, over $F_3$, is never zero over $f_0, 1, g_{\text{full}}$.

Fix: Ok fine. Work with $\overline{c} + f_0, 1, g_{\text{full}}$ for some random $\overline{c}_{2 \times F_n}$.

Issue 3: Even with $\overline{c} + f_0, 1, g_{\text{full}}$ the matrix is still slightly fat and the Vandermonde is slightly tall.

Fix: Prove a really good rank lower bound on the left matrix. (Barely manage to work for a specific explicit polynomial. Phew!)
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Summary

**Theorem**

There is a polynomial $f \in VNP$ such that, for every finite field $\mathbb{F}_q$, any hom. $\Sigma \Pi \Sigma \Pi \Sigma$ circuit computing $f$ over $\mathbb{F}_q$ must have size $\exp(\Omega_q(\sqrt{d}))$. 

Remarks and open problems:


Delicate analysis.

▶ The proof ought to work for IMM also but we don’t have a tight enough analysis (yet).

▶ After this, [Kumar-S] did manage to separate depth-4 and depth-5 in the low-degree regime, but via a different complexity measure.

▶ Other fields?
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