Separating quantum communication and approximate rank

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Roadmap

1. Some background

2. Separating quantum communication and approximate rank
Models of query complexity

- For a function $F$, Randomized (two-sided error of $\varepsilon$) query complexity $R_{\varepsilon dt}(F)$, Quantum (two sided error of $\varepsilon$) query complexity $Q_{\varepsilon dt}(F)$.
- Quadratic separation: using Grover’s search algorithm [Grov95] and its variant proved in [BBHT96].
- OR: $\{0, 1\}^n \rightarrow \{0, 1\}$ outputs 1 if the input contains at least one 1.

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<th>$Q_{1/3}^{dt}$</th>
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<td>$R_{1/3}^{dt}$</td>
<td>2 [BBHT96]</td>
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For a function $F$, approximate polynomial degree $\deg_{\varepsilon}(F)$ is the minimum among the degrees of all polynomials $p(x)$ satisfying $|p(x) - F(x)| \leq \varepsilon$, for all $x$.

It lower bounds quantum query complexity [Beals, Buhrman, Cleve, Mosca, de Wolf 1998]: $Q_{\varepsilon}^{dt}(F) \geq \frac{1}{2}\deg_{\varepsilon}(F)$.

Example: $\deg_{1/3}(OR) = \Theta(\sqrt{n})$.

Other well known bounds: Adversary bound [Ambainis 2000], Negative weights adversary bound [Hoyer, Lee, Spalek 2005].
It is known that $Q^{dt}_{1/3}(F) = O(\deg_{1/3}(F))^6$.
Moreover, there exists a function $F$, such that $Q^{dt}_{1/3}(F) = \Theta(\deg_{1/3}(F))^{1.3219}$ [Ambainis 2003].
Is this the best possible separation?
Cheat sheets

- Aaronson, Ben-David and Kothari [2016] introduced the technique of cheat sheet.
- Follow up to the works Göös, Pitassi and Watson [2015] and Ambainis, Balodis, Belovs, Lee, Santha and Smotrovs [2015].
- A transformation from $F \rightarrow F_{cs}$.

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A transformation from $F \rightarrow F_{cs}$.

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- $F_{cs}$ has two components: ‘c’ copies of a parent function $F$ and a cheat sheet $cs$.
- Compute based on inputs to functions and content at ‘decimal($b$)’.

$$b = F_1, \ldots F_c$$

Diagram:

```
  F_1  •  •  •  F_c
```

```
1  2  •  •  •  2^c
```
Randomized communication complexity $R_{1/3}(F)$: number of bits communicated in a randomized protocol.

Quantum communication complexity $Q_{1/3}(F)$: number of qubits communicated in an entanglement assisted quantum protocol.
Approximate rank for $F$,
$$\text{rk}_\varepsilon(F) = \min_{F'} \{ \text{rk}(F') : |F'(x, y) - F(x, y)| \leq \varepsilon \}.$$

Lower bound on quantum communication complexity [Buhrman and de Wolf 2001, Lee and Shraibman 2008]: For $F : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$,
$$Q_{1/3}(F) \geq \Omega(\log \text{rk}_{1/3}(F) - \log n).$$

Quantum log-rank conjecture: are $Q_{1/3}(F)$ and $\log \text{rk}_{1/3}(M_F)$ polynomially related?
Lower bound on quantum communication complexity

- Approximate rank for $F$,
  $$\text{rk}_\varepsilon(F) = \min_{F'}\{\text{rk}(F') : |F'(x, y) - F(x, y)| \leq \varepsilon\}.$$  

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- Quantum log-rank conjecture: are $Q_{1/3}(F)$ and $\log \text{rk}_{1/3}(F)$ polynomially related?

- Other lower bound: quantum information complexity ([Touchette 2015]).
Notion of cheat sheet extended to communication complexity in A., Belovs, Ben-David, Göös, Jain, Kothari, Lee and Santha [2016].

A similar transformation: $F \rightarrow F_G$, called look-up function.

Super-quadratic separation between $R_{1/3}(F)$ and $Q_{1/3}(F)$. 
Look-up function $F_G$

$F : \mathcal{X} \otimes \mathcal{Y} \rightarrow \{0, 1\}$

$F_1, F_2 \ldots F_c \equiv F$

$G : \mathcal{X}^c \otimes \mathcal{Y}^c \otimes W \rightarrow \{0, 1\}$

$W$ is set of strings

$F_1$

$\begin{array}{c}
F_1 \\
\cdot \\
\cdot \\
\cdot \\
F_c \\
\cdot \\
x_c & y_c \\
u_0 & v_0 \\
u_1 & v_1 \\
u_2 & v_2 \\
\end{array}$

$u_0, v_0, u_1, v_1 \ldots u_2^c, v_2^c \in W$
Look-up function $F_g$

compute $b = (F_1, F_2, \ldots, F_c)$

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Separations in communication complexity

July 8, 2017 14 / 22
Look-up function $F_g$

\[
F_1 \\
\cdot \\
\cdot \\
F_c \\
x_1 \quad y_1 \\
\cdot \\
\cdot \\
x_c \quad y_c \\
\cdot \\
\cdot \\
u_0 \quad v_0 \\
u_1 \quad v_1 \\
u_2 \quad v_2 \\
goto \text{block number decimal}(b)
\]
Look-up function $F_G$

$F_G = 1$

Iff $G(u_b \oplus v_b, x_1, y_1 \ldots x_c, y_c) = 1$
For reasonably non-trivial function $G$, we show the following.

**Theorem**

$$Q_{1/3}(F_G) = \Omega(\log \frac{1}{\text{disc}(F)}).$$

- $\text{disc}(F)$ is the discrepancy of $F$. 
An outline of proof

- We show that for any $r$-round protocol $\Pi$ for $F_G$ that makes an error of $\frac{1}{3}$, there exists a protocol $\Pi'$ for $F$ that makes an error of $\frac{1}{2} - \frac{1}{r^2}$ and communicates the same as in $\Pi$.
- So, $Q_{1/3}(F_G) = \Omega(Q_{\frac{1}{2} - \frac{1}{r^2}}(F)) = \Omega(\log \frac{1}{\text{disc}(F)} - \log r^2)$.
An outline of proof

- Key idea: Quantum cut and paste theorem [Jain, Radhakrishnan and Sen 2003, Nayak and Touchette 2016].
- In a protocol where each player has low information about content of the correct location of other player’s ‘look up part’, output cannot be correct.
Recall: in cheat sheet of Aaronson, Ben-David and Kothari, correct cheat sheet location must certify the evaluation of $F_1, F_2, \ldots F_c$ on their inputs.

Fix a circuit $C$ for $F$, with number of gates $\text{size}(F)$.

We require that $u_b \oplus v_b$ certifies the evaluation of inputs (to $F_1, F_2, \ldots F_c$) on $C$. 
Choice of $G$

**Theorem**

For $G$ as defined above, $\log r_{1/3}(F_G) = O(\sqrt{\text{size}(F)})$.

- Now choose $F$ to be inner product function
  $\text{IP}_n(x, y) = \sum_i x_i y_i \mod 2$. We have $\text{size}(\text{IP}_n) = O(n)$ and
  $\log \frac{1}{\text{disc}(\text{IP}_n)} = \Theta(n)$.

**Theorem**

There exists a total function $F$ such that $Q(F) = \tilde{\Omega}(\log r_{1/3}(F))^2$. 

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July 8, 2017 21 / 22
Open questions

- Can the round dependence in our main result be removed or weakened?
- Is there a general lifting theorem from quantum query complexity to quantum communication complexity?
  - Recently, a lifting theorem shown from randomized query complexity to randomized communication complexity [GPW17].
- Quantum log-rank conjecture?