On algebraic branching programs of small width

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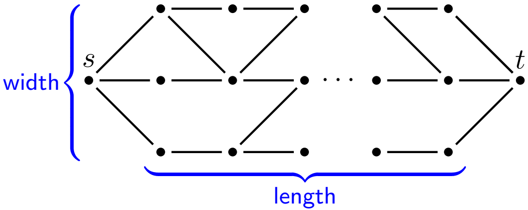
Small width algebraic branching programs: surprisingly powerful

1. Width-2 algebraic branching programs with approximation are as powerful as formulas

2. Width-1 algebraic branching programs with nondeterminism are as powerful as circuits

1. Definitions
   - Algebraic branching programs
   - Formulas
   - Complexity classes $\mathbf{VP}_k$ and $\mathbf{VP}_\mathrm{e}$
   - Approximation classes $\overline{\mathbf{VP}_k}$ and $\overline{\mathbf{VP}_\mathrm{e}}$
2. Historical context
3. Statement of main result
4. Proof sketch
5. Statement of nondeterminism result

Algebraic branching program (ABP) definition



edge labels are affine linear forms:
$\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$
$(\alpha_i \in \mathbb{C})$

$$f(x_1, \ldots, x_n) = \sum_{\substack{s\text{-}t \text{ paths} \\ \text{in graph}}} \text{product of edge labels on path}$$

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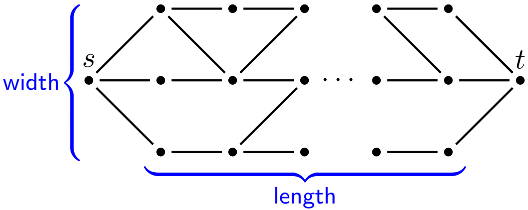

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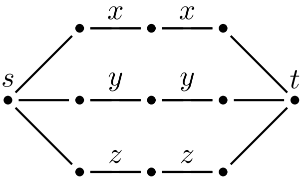
Example

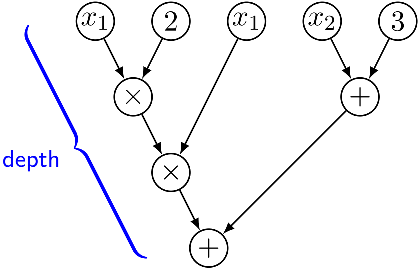
$$x^2 + y^2 + z^2 = \sum_{\substack{s\text{-}t \text{ path} \\ \text{products}}}$$



Complexity

$L_k(f)$ = minimum length of any width-k ABP computing f

Formula definition



size = number of nodes

leaves

variables x_i

constants $\alpha_i \in \mathbb{C}$

nodes

$+$, $\times$

fan-in 2

fan-out 1

$f(x_1, \dots, x_n)$ = evaluation of tree

Complexity

$L_\mathrm{e}(f)$ = minimum size of any formula computing f

Classes $\mathbf{VP}_k$ and $\mathbf{VP}_\mathrm{e}$ definition

Recall:
- L_k = width-k ABP length
- L_e = formula size

family: sequence $(f_n)_{n\in\mathbb{N}}$ of polynomials $f_n(x_1,\dots,x_{\mathrm{poly}(n)})$

$$\mathbf{VP}_k \coloneqq \big\{\text{families } (f_n)_{n\in\mathbb{N}} \text{ with } L_k(f_n)=\mathrm{poly}(n)\big\} \qquad k\in\mathbb{N}$$

$$\mathbf{VP}_\mathrm{e} \coloneqq \big\{\text{families } (f_n)_{n\in\mathbb{N}} \text{ with } L_\mathrm{e}(f_n)=\mathrm{poly}(n)\big\}$$

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Ben-Or and Cleve (1988) inspired by Barrington's theorem (1986)

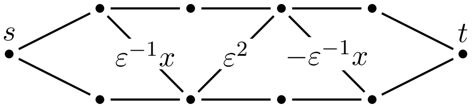
$\mathbf{VP}_3 = \mathbf{VP}_4 = \cdots = \mathbf{VP}_\mathrm{e}$
In particular: width-3 ABPs can compute any polynomial

Allender and Wang (2011)

Strict inclusion: $\mathbf{VP}_2 \subsetneq \mathbf{VP}_3$
No width-2 ABP computes $\quad x_1x_2 + \cdots + x_{15}x_{16}$

Approximation

$$\sum_{s\text{-}t \text{ path products}}$$



$$= 1 + 1 + \varepsilon^{-1}x - \varepsilon^{-1}x + \varepsilon x - \varepsilon x - x^2 + \varepsilon^2$$

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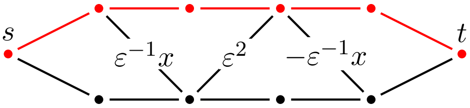

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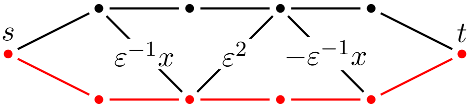

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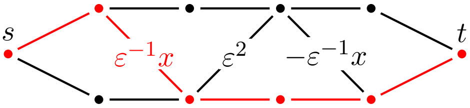

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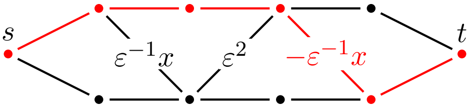
$\sum_{s\text{-}t \text{ path products}}$ 

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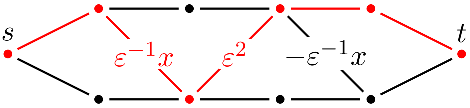

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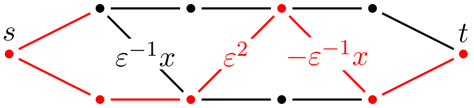

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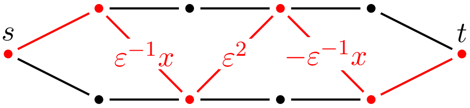
$\sum_{s\text{-}t \text{ path products}}$ 

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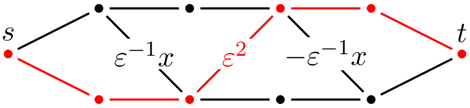

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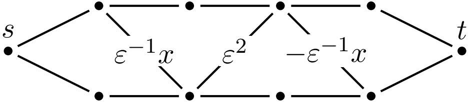

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$$= 2 - x^2 + \varepsilon^2$$

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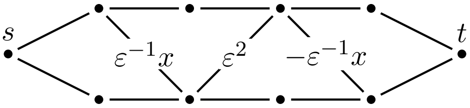

$$= 1 + 1 + \varepsilon^{-1}x - \varepsilon^{-1}x + \varepsilon x - \varepsilon x - x^2 + \varepsilon^2$$
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- $2 - x^2 + \varepsilon^2 \xrightarrow{\varepsilon \to 0} 2 - x^2$
- $L_2(2 - x^2 + \varepsilon^2) \leq 4 \quad (\varepsilon > 0)$

We say "$\overline{L_2}(2 - x^2) \leq 4$"

Approximation

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Border complexity   cp. border rank (Bini et al., Strassen)

$V = \mathbb{C}[x_1, \dots, x_n]_{\leq d}$ degree $\leq d$ polyn. endowed with Euclidean norm

$\overline{L}(f) :=$ smallest r for which there exist $(g_\varepsilon)_{\varepsilon \in \mathbb{R}_{>0}} \subseteq V$ and

- $\lim_{\varepsilon \to 0} g_\varepsilon = f$
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$\overline{\mathbf{VP}_k} = \{\text{families } (f_n)_{n \in \mathbb{N}} \text{ with } \overline{L_k}(f_n) = \mathrm{poly}(n)\}$ $k \in \mathbb{N}$

$\overline{\mathbf{VP}_\mathrm{e}} = \{\text{families } (f_n)_{n \in \mathbb{N}} \text{ with } \overline{L_\mathrm{e}}(f_n) = \mathrm{poly}(n)\}$

Clearly $\overline{L}(f) \leq L(f)$. Therefore $\mathbf{VP}_k \subseteq \overline{\mathbf{VP}_k}$, $\mathbf{VP}_\mathrm{e} \subseteq \overline{\mathbf{VP}_\mathrm{e}}$, etc

More historical context

Valiant (1979) $\mathbf{VP_e} \subseteq \mathbf{VP_s} \subseteq \mathbf{VP} \quad \subseteq \quad \mathbf{VNP}$

Valiant's conjectures $\mathbf{VP_e}, \mathbf{VP_s}, \mathbf{VP} \quad \overset{?}{\not\supseteq} \quad \mathbf{VNP}$

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Strassen, Mulmuley-Sohoni (GCT), Bürgisser

Extended conjectures $\overline{\mathbf{VP_s}}, \overline{\mathbf{VP}} \quad \overset{?}{\not\supseteq} \quad \mathbf{VNP}$

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Extended conjectures $\quad \overline{\mathbf{VP_s}}, \overline{\mathbf{VP}} \quad \overset{?}{\not\supseteq} \quad \mathbf{VNP}$

Proving e.g. $\mathbf{VP_e} \not\supseteq \mathbf{VNP}$ using any geometric technique
(e.g. shifted partial derivatives or geometric complexity theory)
automatically implies $\overline{\mathbf{VP_e}} \not\supseteq \mathbf{VNP}$.

We study $\quad \overline{\mathbf{VP_e}}$

Recent work on closures of classes:
Forbes (2016), Grochow-Mulmuley-Qiao (2016)

Statement of main result

Main theorem: $\overline{\mathbf{VP}_2} = \overline{\mathbf{VP}_\mathrm{e}}$

$$\begin{array}{ccccc} \overline{\mathbf{VP}_2} & = & \overline{\mathbf{VP}_3} & = & \overline{\mathbf{VP}_\mathrm{e}} \\ \cup\!\!\!\!\not\,| & & \cup\!| & & \cup\!| \\ \mathbf{VP}_2 & \subsetneq & \mathbf{VP}_3 & = & \mathbf{VP}_\mathrm{e} \end{array}$$

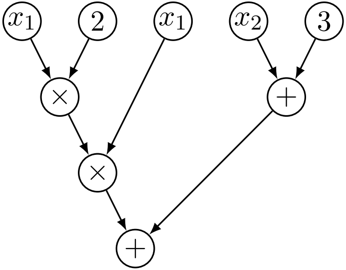
Ben-Or–Cleve

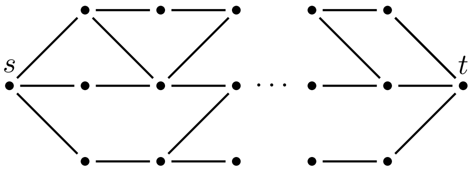
Allender–Wang

Corollary: strict inclusion $\mathbf{VP}_2 \subsetneq \overline{\mathbf{VP}_2}$

Ben-Or and Cleve construction

To prove: $\mathbf{VP}_{\mathrm{e}} \subseteq \mathbf{VP}_3$





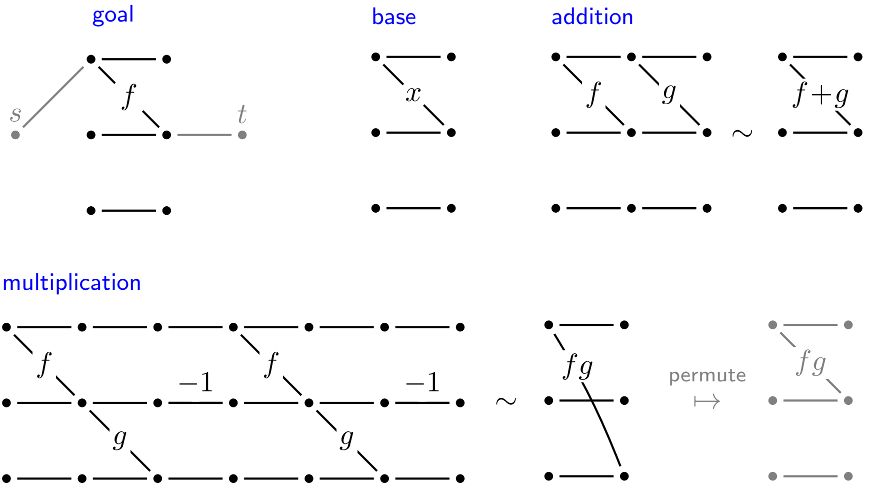
edge labels: affine linear forms

size s formula $\rightsquigarrow$ size $\mathrm{poly}(s)$ width-3 ABP

Brent (1974) depth reduction:

size $\mathrm{poly}(s)$
depth $\mathcal{O}(\log s)$ formula

To prove: $\mathbf{VP}_{\mathrm{e}} \subseteq \mathbf{VP}_3$



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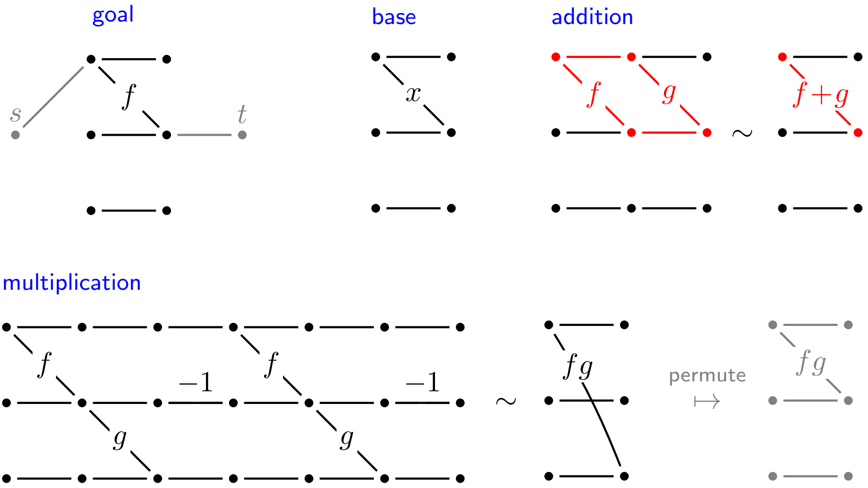

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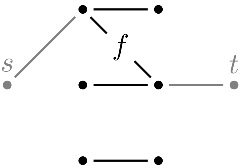
goal



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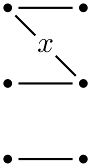
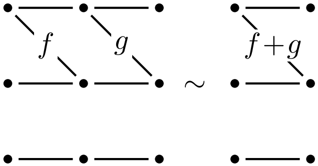
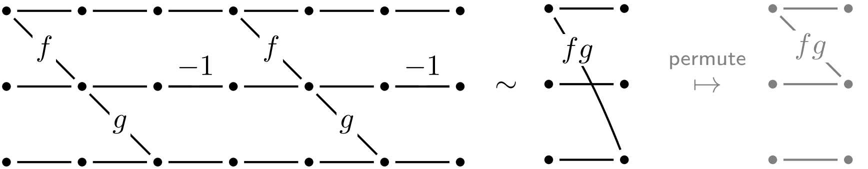


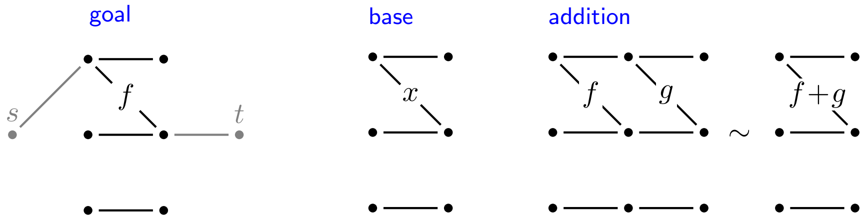
addition

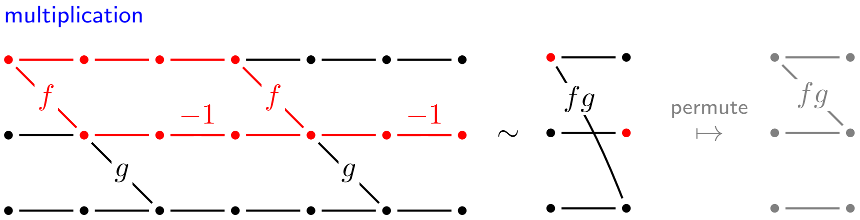


multiplication



To prove: $\mathbf{VP}_e \subseteq \mathbf{VP}_3$





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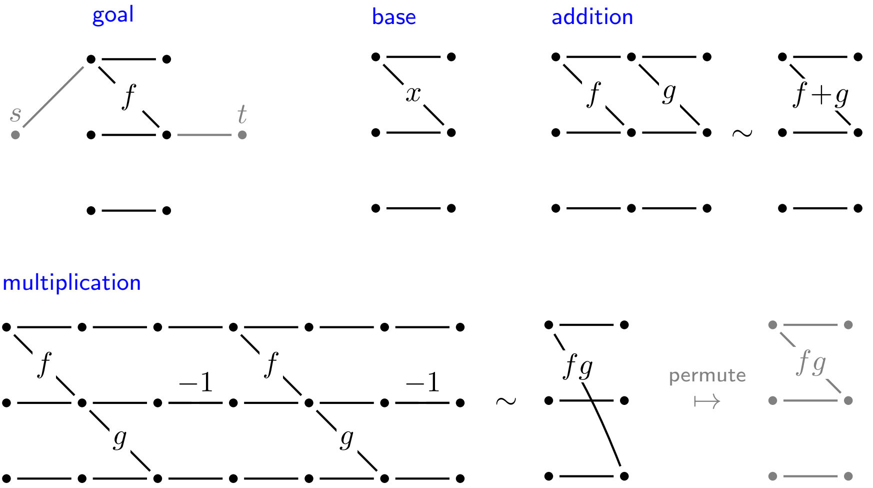

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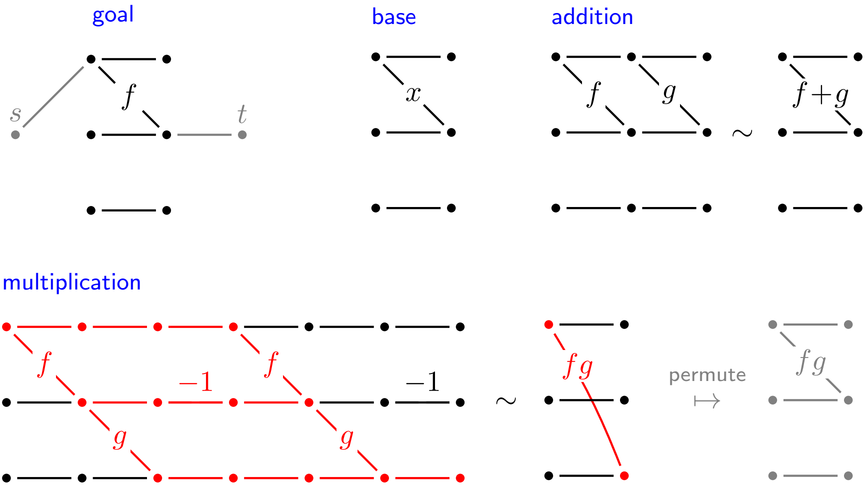

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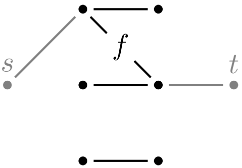

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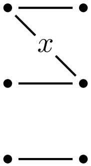

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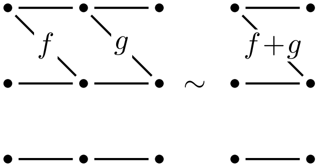

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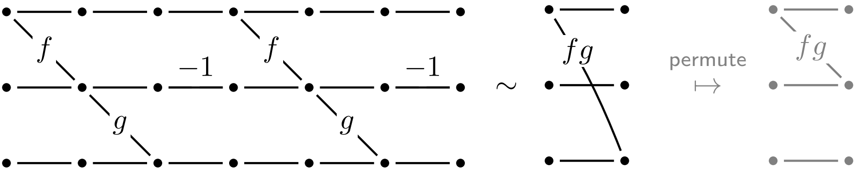

Our construction

To prove: $\mathbf{VP}_{\mathrm{e}} \subseteq \overline{\mathbf{VP}_2}$ (then $\overline{\mathbf{VP}_{\mathrm{e}}} \subseteq \overline{\mathbf{VP}_2}$ follows)

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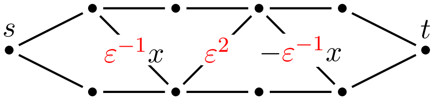
Recall: computational model

$\sum$ over s-t path products



$$= 2 + x^2 + \varepsilon \quad \xrightarrow{\varepsilon \to 0} \quad 2 + x^2$$

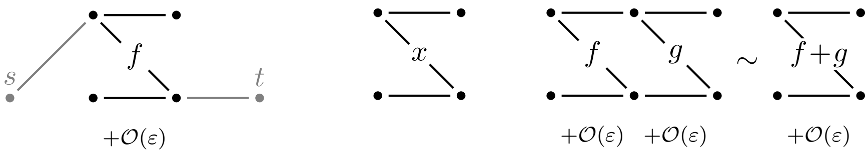
We need

$$= f + \underbrace{\varepsilon f_1 + \varepsilon^2 f_2 + \cdots}_{\mathcal{O}(\varepsilon)} \quad \xrightarrow{\varepsilon \to 0} \quad f$$

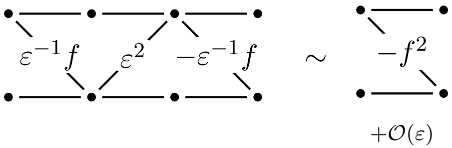
Our construction

To prove: $\mathbf{VP}_\mathrm{e} \subseteq \overline{\mathbf{VP}_2}$

goal — base — addition



squaring (idea)



multiplication $\quad fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$

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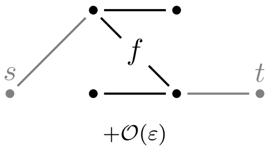

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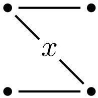

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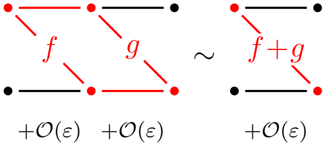

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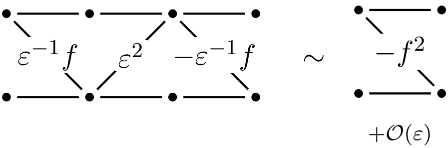

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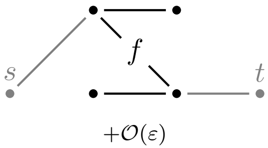

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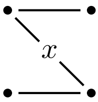
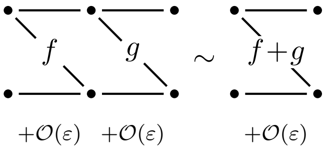
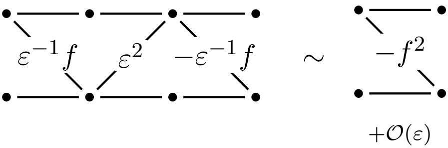


addition



squaring (idea)

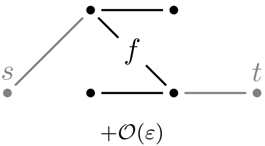


multiplication $\quad fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$

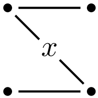
Our construction

To prove: $\mathbf{VP}_\mathrm{e} \subseteq \overline{\mathbf{VP}_2}$

goal



base



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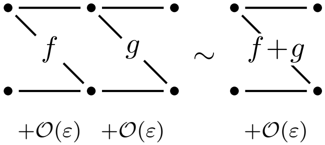

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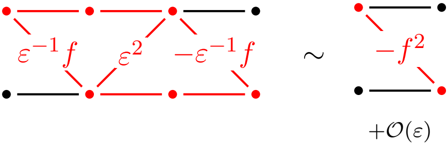

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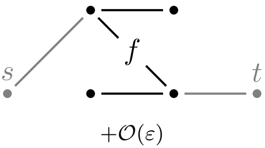

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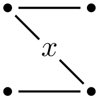

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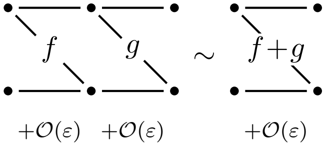

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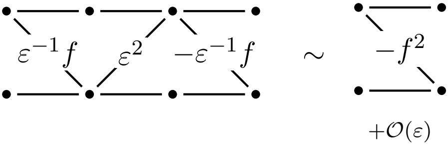

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Statement of nondeterminism result

Recall: $(g_n) \in \mathbf{VP}_1$ means g_n is product of $\mathrm{poly}(n)$ many affine linear forms

Definition: $(f_n) \in \mathbf{VNP}_1$ if

- $\exists\, (g_n) \in \mathbf{VP}_1$
- $f_n(x_1, \ldots, x_{p(n)}) = \sum_{b \in \{0,1\}^{\mathrm{poly}(n)}} g_n(x_1, \ldots, x_{p(n)}, b_1, \ldots, b_{\mathrm{poly}(n)})$

Naturally generalises to $\mathbf{VNP}_\mathrm{e}$ and $\mathbf{VNP}$

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Valiant (1980): $\mathbf{VNP}_\mathrm{e} = \mathbf{VNP}$

Theorem: $\mathbf{VNP}_1 = \mathbf{VNP}$

Corollary: strict inclusions $\mathbf{VP}_1 \subsetneq \mathbf{VNP}_1$ and $\mathbf{VP}_2 \subsetneq \mathbf{VNP}_2$

$$
\begin{array}{ccccccc}
\overline{\mathbf{VP}_1} & \subsetneq & \boxed{\overline{\mathbf{VP}_2} \quad = \quad \overline{\mathbf{VP}_{\mathrm{e}}}} & & & \subseteq & \overline{\mathbf{VP}} \\
\| & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathbf{VP}_1 & \subsetneq & \mathbf{VP}_2 & \subsetneq & \mathbf{VP}_{\mathrm{e}} & \subseteq & \mathbf{VP} \\
\rotatebox{-90}{$\subsetneq$} & & \rotatebox{-90}{$\subsetneq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\
\boxed{\mathbf{VNP}_1 \; = \; \mathbf{VNP}_2 \; = \; \mathbf{VNP}_{\mathrm{e}}} & & & & & = & \mathbf{VNP}
\end{array}
$$

$$
\begin{array}{ccccccc}
\overline{\mathbf{VP}_1} & \subsetneq & \overline{\mathbf{VP}_2} & = & \overline{\mathbf{VP}_\mathrm{e}} & \subseteq & \overline{\mathbf{VP}} \\
\rotatebox{90}{$=$} & & \rotatebox{-90}{$\supsetneq$} & & \rotatebox{-90}{$\supseteq$} & & \rotatebox{-90}{$\supseteq$} \\
\mathbf{VP}_1 & \subsetneq & \mathbf{VP}_2 & \subsetneq & \mathbf{VP}_\mathrm{e} & \subseteq & \mathbf{VP} \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathbf{VNP}_1 & = & \mathbf{VNP}_2 & = & \mathbf{VNP}_\mathrm{e} & = & \mathbf{VNP}
\end{array}
$$

Thank you!

Proof sketch $\mathbf{VNP}_1 = \mathbf{VNP}$

1. We know $\mathbf{VP}_{\mathrm{e}} \subseteq \mathbf{VP}_3$ (Ben-Or–Cleve).
2. We prove $\mathbf{VP}_3 \subseteq \mathbf{VNP}_1$. Construction: let nondeterminism select s-t paths in width-3 ABP.
3. This shows $\mathbf{VP}_{\mathrm{e}} \subseteq \mathbf{VNP}_1$. This implies $\mathbf{VNP}_{\mathrm{e}} \subseteq \mathbf{VNP}_1$. We know $\mathbf{VNP} = \mathbf{VNP}_{\mathrm{e}}$ (Valiant).

Side result: the continuant

Definition continuant

$$F_0 = 1$$
$$F_1(x_1) = x_1$$
$$F_n(x_1, \ldots, x_n) = x_n \cdot F_{n-1}(x_1, \ldots, x_{n-1}) + F_{n-2}(x_1, \ldots, x_{n-2})$$

Example: $F_n(1, 1, \ldots, 1) = n$th Fibonacci number

Continuant complexity
$L_F(f) =$ smallest n such that $f(x_1, \ldots, x_n) = F_n(\ell_1, \ldots, \ell_n)$
L_F induces classes $\mathbf{VP}_F$ and $\overline{\mathbf{VP}_F}$

Proposition: $\overline{\mathbf{VP}_F} = \overline{\mathbf{VP}_\mathrm{e}}$

$$
\begin{array}{ccccccccccccccccc}
\overline{\mathbf{VNP}_1^{\mathrm{wst}}} & \subsetneq & \overline{\mathbf{VNP}_1^{\mathrm{w}}} & \subsetneq & \overline{\mathbf{VNP}_1^{\mathrm{g}}} & = & \overline{\mathbf{VNP}_2^{\mathrm{wst}}} & = & \overline{\mathbf{VNP}_2^{\mathrm{w}}} & = & \overline{\mathbf{VNP}_2^{\mathrm{g}}} & = & \overline{\mathbf{VNP}_{\mathrm{e}}} & = & \overline{\mathbf{VNP}_{\mathrm{s}}} & = & \overline{\mathbf{VNP}} \\
\| & & \| & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathbf{VNP}_1^{\mathrm{wst}} & \subsetneq & \mathbf{VNP}_1^{\mathrm{w}} & \subsetneq & \mathbf{VNP}_1^{\mathrm{g}} & = & \mathbf{VNP}_2^{\mathrm{wst}} & = & \mathbf{VNP}_2^{\mathrm{w}} & = & \mathbf{VNP}_2^{\mathrm{g}} & = & \mathbf{VNP}_{\mathrm{e}} & = & \mathbf{VNP}_{\mathrm{s}} & = & \mathbf{VNP} \\
\| & & \| & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\overline{\mathbf{VP}_1^{\mathrm{wst}}} & \subsetneq & \overline{\mathbf{VP}_1^{\mathrm{w}}} & \subsetneq & \overline{\mathbf{VP}_1^{\mathrm{g}}} & \subsetneq & \overline{\mathbf{VP}_2^{\mathrm{wst}}} & = & \overline{\mathbf{VP}_2^{\mathrm{w}}} & = & \overline{\mathbf{VP}_2^{\mathrm{g}}} & = & \overline{\mathbf{VP}_{\mathrm{e}}} & \subseteq & \overline{\mathbf{VP}_{\mathrm{s}}} & \subseteq & \overline{\mathbf{VP}} \\
\| & & \| & & \| & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\overline{\mathbf{VP}_1^{\mathrm{wst}}}^{\mathrm{poly}} & \subsetneq & \overline{\mathbf{VP}_1^{\mathrm{w}}}^{\mathrm{poly}} & \subsetneq & \overline{\mathbf{VP}_1^{\mathrm{g}}}^{\mathrm{poly}} & \subsetneq & \overline{\mathbf{VP}_2^{\mathrm{wst}}}^{\mathrm{poly}} & = & \overline{\mathbf{VP}_2^{\mathrm{w}}}^{\mathrm{poly}} & = & \overline{\mathbf{VP}_2^{\mathrm{g}}}^{\mathrm{poly}} & = & \overline{\mathbf{VP}_{\mathrm{e}}}^{\mathrm{poly}} & \subseteq & \overline{\mathbf{VP}_{\mathrm{s}}}^{\mathrm{poly}} & \subseteq & \overline{\mathbf{VP}}^{\mathrm{poly}} \\
\| & & \| & & \| & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \| & & \| & & \| \\
\mathbf{VP}_1^{\mathrm{wst}} & \subsetneq & \mathbf{VP}_1^{\mathrm{w}} & \subsetneq & \mathbf{VP}_1^{\mathrm{g}} & \subsetneq & \mathbf{VP}_2^{\mathrm{wst}} & \subseteq & \mathbf{VP}_2^{\mathrm{w}} & \subsetneq & \mathbf{VP}_2^{\mathrm{g}} & \subsetneq & \mathbf{VP}_{\mathrm{e}} & \subseteq & \mathbf{VP}_{\mathrm{s}} & \subseteq & \mathbf{VP}
\end{array}
$$