Noise Stability is Low-Dimensional

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Gaussian noise stability

Take $X$ and $Y$ a pair of $\rho$-correlated Gaussians in $\mathbb{R}^n$ ($0 < \rho < 1$).

For a partition $A = (A_1, \ldots, A_k)$ of $\mathbb{R}^n$ into $k$ parts, define

$$\text{Stab}_\rho(A) = \Pr(X \text{ and } Y \text{ land in the same part})$$
Gaussian noise stability

Noise stable

Not noise stable
Theorem (Borell ’85)
For a partition of $\mathbb{R}^n$ into two parts of equal Gaussian measure,

$$\text{Stab}_\rho(A) \leq \frac{1}{2} + \frac{\sin^{-1} \rho}{\pi}.$$ 

Equality is attained for a partition into half-spaces.
Gaussian noise stability

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- one-dimensional phenomenon
Gaussian noise stability

Theorem (???)

For a partition of $\mathbb{R}^n$ into three parts of equal Gaussian measure,
Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.
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Conjecture (Multi-dimensional peace sign conjecture)
*For partitions into k equal measures, the optimal partition occurs in \( \mathbb{R}^{k-1} \). It looks like a multi-dimensional peace sign.*
Theorem (De-Mossel-N.)

For any $k$ and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an $\epsilon$-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.
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Corollary
The optimal value of $k$-part noise stability is computable.
Gaussian noise stability

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**Corollary**

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**Corollary (sort of)**

The non-interactive correlation distillation value with $k$-ary outputs is computable.
Correlation distillation

Goal: produce uniform output, agree with maximal probability.

What is the probability of agreement?
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Ghazi-Kamath-Sudan ’16: reduction to correlated Gaussian signals
The main theorem

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Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ ($n$ huge) and try to “simulate” it in $\mathbb{R}^{n_0}$. 
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1. An optimal partition is close to a bounded-degree polynomial threshold function (PTF)
Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ ($n$ huge) and try to "simulate" it in $\mathbb{R}^{n_0}$.

1. An optimal partition is close to a bounded-degree polynomial threshold function (PTF)

2. A bounded-degree PTF can be approximately simulated by a bounded-degree PTF on a bounded number of variables
Step 1: approximation by polynomials
Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \rightarrow \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$. 
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Hermite expansion $f(x) = \sum_{\alpha, i} \hat{f}_{\alpha, i} H_{\alpha}(x)e_i$
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Facts: $1 = \sum_{\alpha, i} \hat{f}_{\alpha, i}^2$ and $\text{Stab}_\rho(f) = \sum_{\alpha, i} \rho^{\deg(H_{\alpha})} \hat{f}_{\alpha, i}^2$
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Idea: noise stability $\Rightarrow$ lots of “low-degree” weight

$\Rightarrow$ approximate $f$ by truncating the expansion

Real proof goes through a smoothing/rounding procedure, and a connection with Gaussian surface area (KNOW '15).
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Hermite expansion

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Step 2: dimension reduction
Polynomial structure theorem (De-Servedio)

\[ p_1(\overline{x}) \ p_2(\overline{x}) \ \ldots \ p_k(\overline{x}) \]

\[ \underleftarrow{v} \ \underleftarrow{v} \ \underleftarrow{v} \]

\[ q_1(\overline{v}) \ q_2(\overline{v}) \ \ldots \ q_k(\overline{v}) \]

\[ v_1(\overline{x}) \ v_2(\overline{x}) \ v_3(\overline{x}) \ \ldots \ v_\ell(\overline{x}) \]

\[ x_1 \ x_2 \ x_3 \ x_4 \ \ldots \ x_n \]

where \( \ell \) is bounded and \( v_1, \ldots, v_\ell \) are "nice"
Polynomial Central Limit Theorem (De-Servedio)

\[ p_1(\overline{x}) \quad p_2(\overline{x}) \quad \ldots \quad p_k(\overline{x}) \]

\[ q_1(\overline{u}) \quad q_2(\overline{u}) \quad \ldots \quad q_k(\overline{u}) \]

\[ v_1(\overline{x}) \quad v_2(\overline{x}) \quad v_3(\overline{x}) \quad \ldots \quad v_\ell(\overline{x}) \]

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_n \]
“Nice” polynomials satisfy a CLT, so they may as well be linear functions of \( \ell \) variables.
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Conjecture (Peace sign conjecture)

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Thank you!