Proof Complexity Lower Bounds from Algebraic Circuit Complexity

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Subset Sum

Question (Subset Sum)

Given $\alpha_1, \ldots, \alpha_n, \beta \in \mathbb{C}$*, prove there is* **no** *subset* $S \subseteq [n]$ *with the sum* $\sum_{i \in S} \alpha_i = \beta$*? Equivalently, prove there are* **no** *solutions to*

$$0 = x_1^2 - x_1 = \cdots = x_n^2 - x_n = \alpha_1 x_1 + \cdots + \alpha_n x_n - \beta \ .$$

Is coNP-hard, NP $\neq$ coNP $\implies$ any proof must be long

goal: prove *unconditional* lower bounds on lengths of proofs in strong *algebraic* proof systems.

Nullstellensatz Proofs (i)

Let $\overline{f} := (f_1, \dots, f_m)$ be a system of polynomials in $\mathbb{C}[x_1, \dots, x_n]$.

Theorem (Hilbert's Nullstellensatz)

The system $f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0$ *has no solution iff there are* $g_1, \dots, g_m \in \mathbb{C}[\overline{x}]$ *such that*

$$g_1(\overline{x}) \cdot f_1(\overline{x}) + \cdots + g_m(\overline{x}) \cdot f_m(\overline{x}) = 1 \ .$$

Gives a *sound* and *complete* proof system for unsatisfiability.
complexity: only weak bounds for $\overline{g}$ in general, ex: simple unsatisfiable $\overline{f}$ can require $\deg \overline{g} \geq \exp(m)$.
but: coNP-statements concern $\overline{x} \in \{0,1\}^n$ — polynomials over $\{0,1\}^n$ are degree $\leq n$.

Nullstellensatz Proofs (ii)

$\overline{f} := (f_1, \ldots, f_m)$, $\overline{x}^2 - \overline{x} := (x_1^2 - x_1, \ldots, x_n^2 - x_n)$.

Theorem (Boolean Nullstellensatz)

The system $\overline{f} = \overline{0}$ has no solution over $\overline{x} \in \{0,1\}^n$ iff the system $\overline{f}, \overline{x}^2 - \overline{x}$ is unsatisfiable iff there are $g_1, \ldots, g_m, h_1, \ldots, h_n \in \mathbb{C}[\overline{x}]$ such that

$$\sum_j g_j(\overline{x}) \cdot f_j(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1 \; .$$

complexity: $\deg \overline{g}, \overline{h} \leq O(n)$, $\overline{g}, \overline{h}$ have at most $2^{O(n)}$ monomials

prior work ([BIK+96a, CEI96, BIK+96b, Raz98, Gri98, IPS99, BGIP01, AR01,...]): exhibit simple $\overline{f}$ where

- $\deg \overline{g}, \overline{h} \geq \Omega(n)$
- $\overline{g}, \overline{h}$ require $2^{\Omega(n)}$ monomials

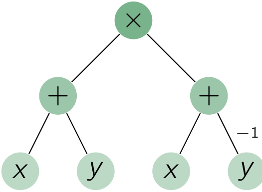
Nullstellensatz Proofs (iii)

Theorem ([GrochowPitassi14])

Let CNF $C = C_1 \wedge \cdots \wedge C_m$ be unsatisfiable and be encoded by the equations $f_1, \ldots, f_m, \overline{x}^2 - \overline{x}$. Then there are $\overline{g}, \overline{h}$ such that $\sum_j g_j \cdot f_j + \sum_i h_i \cdot (x_i^2 - x_i) = 1$, where

- *If there is a size-s Frege proof that C is unsatisfiable, then there are $\overline{g}, \overline{h}$ with $\text{poly}(n, m, s)$-size algebraic formulas.*
- *There are $\overline{g}, \overline{h}$ in $\text{VNP} \approx \{$explicit polynomials$\}$.*

Algebraic formulas are a succinct model of computation for polynomials, e.g. $x^2 - y^2 = (x + y)(x - y)$ can be given by



The Ideal Proof System (IPS)

$\overline{f} := (f_1, \ldots, f_m)$, $\overline{x}^2 - \overline{x} := (x_1^2 - x_1, \ldots, x_n^2 - x_n)$.

Definition ([GrochowPitassi14])

A size-s **(linear) Ideal Proof System (IPS)** proof of unsatisfiability of $\overline{f}, \overline{x}^2 - \overline{x}$ using $\mathcal{C}$-computations is $\overline{g}, \overline{h}$ where

- $\sum_j g_j(\overline{x}) \cdot f_j(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1$.
- each g_j, h_i is size-s $\mathcal{C}$-formula.

- proof verification: via *Polynomial Identity Testing*, only randomized algorithms known in general.
- [GP14]: formula-IPS is as powerful as Frege.
- [GP14]: lower bounds for $\mathcal{C}$-proofs of CNFs $\implies$ lower bounds for $\mathcal{C}$-computations of the permanent
- [FTL15]: *non-commutative* formula-IPS is equivalent to Frege.

goal: prove lower bounds for $\mathcal{C}$-IPS for interesting $\mathcal{C}$.

Multilinear Formulas

Definition

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is **multilinear** if the individual degree of each variable x_i is at most 1, that is

$$f(\overline{x}) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i \ .$$

A formula is multilinear if each gate is multilinear.

- A multilinear polynomial is uniquely determined by evaluations over $\{0,1\}^n$.
- [Raz04,RY09]: permanent and determinant require $n^{\Omega(\lg n)}$-size multilinear formulas, $2^{n^{\Omega(1)}}$-size constant-depth multilinear formulas
- [RT08]: defined proof system based on multilinear formulas, short proofs for pigeonhole principle, etc.

goal: prove lower bounds for multilinear-formula-IPS.

Our Results

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$, is unsatisfiable subset-sum instance.

Theorem ([ImpagliazzoPudlákSgall99])

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$ *requires Nullstellensatz refutations of*

- *degree* $\geq \Omega(n)$.
- $2^{\Omega(n)}$*-monomials.*

Related to Pigeonhole Principle, well-known "hard" principle.

Theorem (Upper Bounds for Subset-Sum)

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$ *has a* $\mathrm{poly}(n)$*-size* $\mathcal{C}$*-IPS proof for* $\mathcal{C} =$

- *depth-3 multilinear formulas*
- *read-once oblivious algebraic branching programs (roABPs)*

Strengthens related upper bounds of [GH03,RT08].

Our Results (ii)

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j}x_ix_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ requires

- *multilinear-formula-IPS proofs of $n^{\Omega(\lg n)}$-size*
- *constant-depth-multilinear-formula-IPS proofs of $2^{n^{\Omega(1)}}$-size*
- *roABP-IPS proofs of $2^{\Omega(n)}$-size (in every order)*

First such lower bounds, matches much of the frontier of lower bounds in algebraic complexity theory.

Proven via *functional lower bounds*.

Functional Lower Bounds

circuit complexity: single polynomial requires large formulas.
proof complexity: *every* proof requires large formulas.

idea: if "*unique*" proof then only study single polynomial.

Consider an unsatisfiable system $f(\overline{x}), \overline{x}^2 - \overline{x}$, with proof

$$g(\overline{x}) \cdot f(\overline{x}) + \sum_i h_i(\overline{x}) \cdot (x_i^2 - x_i) = 1 \ .$$

$$\begin{aligned} g(\overline{x}) \cdot f(\overline{x}) &= 1, & \overline{x} &\in \{0,1\}^n \\ g(\overline{x}) &= 1/f(\overline{x}), & \overline{x} &\in \{0,1\}^n \end{aligned}$$

$\implies$ g unique as a *function* or as *multilinear* polynomial.

goal: find *easy* $f(\overline{x})$ so <u>any</u> g with $g|_{\{0,1\}^n} = \frac{1}{f}|_{\{0,1\}^n}$ is *hard*.
A type of *functional lower bound* [GR00,**F**KS15].

Our Results (iii)

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ *requires*

- *multilinear-formula-IPS proofs of $n^{\Omega(\lg n)}$-size*
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Proof.

- prove functional lower bound for *degree*.
- "lift" to functional lower bound for *evaluation dimension*.
- conclude functional lower bound for circuit classes via known relations to evaluation dimension.
- conclude IPS lower bounds. □

Functional Lower Bound — Degree

$x_1 + \cdots + x_n + 1, \overline{x}^2 - \overline{x}$.

Proposition

$f \in \mathbb{C}[\overline{x}]$. *If*

$$f(\overline{x}) = \frac{1}{x_1 + \cdots + x_n + 1}, \qquad \overline{x} \in \{0,1\}^n\,,$$

then $\deg f \geq n$.

Tight. Strengthens prior $\deg f \geq {}^{n}/_{2}$ [IPS99].

Proof.

- multilinearize: $f \mapsto \mathrm{ml}(f)$ with $f|_{\{0,1\}^n} = \mathrm{ml}(f)|_{\{0,1\}^n}$, and $\deg f \geq \deg \mathrm{ml}(f)$.
- $\mathrm{ml}(f)$ uniquely determined, compute it explicitly, $\deg \mathrm{ml}(f) = n$. □

Evaluation Dimension

$f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Study interaction between $\overline{x}$ and $\overline{y}$.

Definition ([Nis91,Sap12,FS13b])

The **set of evaluations** of f is

$$\mathbf{Eval}_{\overline{x}|\overline{y}}(f) := \{f(\overline{x}, \overline{\beta})\}_{\overline{\beta} \in \{0,1\}^n} \subseteq \mathbb{C}[\overline{x}] \; .$$

The **evaluation dimension** of f is $\dim_{\mathbb{C}} \mathbf{Eval}_{\overline{x}|\overline{y}}(f)$.

Well-studied complexity measure, used for many lower bounds:

- multilinear formulas [Raz04,RY09,...]
- non-commutative ABPs, roABPs [Nisan91,**F**S13b,...]
- depth-3 powering formulas [Saxena08,**F**S13b,...]

Functional Lower Bound — Evaluation Dimension

$\sum_{i=1}^{n} x_i y_i + 1, \overline{x}^2 - \overline{x}, \overline{y}^2 - \overline{y}$.

Proposition

$f(\overline{x}, \overline{y}) = \frac{1}{\sum_i x_i y_i + 1}$ for $\overline{x}, \overline{y} \in \{0,1\}^n$, then $\dim \mathbf{Eval}_{\overline{x}|\overline{y}}(f) \geq 2^n$.

Proof.

- $\mathbf{Eval}_{\overline{x}|\overline{y}}(f) = \{f(\overline{x}, \overline{\beta})\}_{\overline{\beta} \in \{0,1\}^n}$.
- For $\overline{x} \in \{0,1\}^n$, $f(\overline{x}, \overline{\beta}) = \frac{1}{\sum_i x_i \beta_i + 1} \overset{\overline{\beta} \leftrightarrow S}{=} \frac{1}{\sum_{i \in S} x_i + 1}$
- $\mathrm{ml}(f(\overline{x}, S))$ has degree $\leq |S|$, $\geq |S| \implies$ $\mathrm{ml}(f(\overline{x}, S)) = \prod_{i \in S} x_i + (\text{lower terms})$.
- $\mathrm{ml}(f(\overline{x}, S))$ triangular system $\implies$ linearly independent.

$$\dim \mathbf{Eval}_{\overline{x}|\overline{y}}(f) \geq \dim \mathrm{ml}(\mathbf{Eval}_{\overline{x}|\overline{y}}(f)) = \dim \{\mathrm{ml}(f(\overline{x}, S))\}_{S \subseteq [n]}$$

$$= \dim \left\{ \prod_{i \in S} x_i + (\text{lower terms}) \right\}_S = 2^n . \qquad \square$$

Our Results (iv)

Theorem (Lower Bounds for Subset-Sum Variants)

$\sum_{i<j} z_{i,j}x_ix_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$ requires

- *multilinear-formula-IPS proofs of $n^{\Omega(\lg n)}$-size*
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Proof.

- degree $\geq n$ functional lower bound for $\frac{1}{\sum_i x_i + 1}$
- $\dim \mathbf{Eval}_{\overline{x}|\overline{y}} \geq 2^n$ functional lower bound for $\frac{1}{\sum_i x_i y_i + 1}$
- symmetrize to get functional lower bound for $\frac{1}{\sum_{i<j} z_{i,j}x_ix_j + 1}$
- invoking existing relations to restricted circuit classes
- convert functional lower bound to IPS lower bound □

Conclusions

This talk:

- upper bounds for proving unsatisfiability of $\sum_i x_i + 1, \overline{x}^2 - \overline{x}$
  - depth-3 multilinear formulas
  - read-once oblivious algebraic branching programs (roABPs)
- lower bounds for proving unsatisfiability of $\sum_{i<j} z_{i,j} x_i x_j + 1, \overline{x}^2 - \overline{x}, \overline{z}^2 - \overline{z}$
  - multilinear-formula-IPS proofs of $n^{\Omega(\lg n)}$-size
  - constant-depth-multilinear-formula-IPS proofs of $2^{n^{\Omega(1)}}$-size
  - roABP-IPS proofs of $2^{\Omega(n)}$-size (in every order)

Other results:

- "non-linear" IPS = "linear" IPS
- lower bounds for multiples: if f requires large formulas, does $g \cdot f$ for every non-zero g?

Open Questions:

- Lower bounds for unsatisfiability of $f_1, \ldots, f_m, \overline{x}^2 - \overline{x}$ with $m > 1$?