Invariance principle on the slice

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1. The result
2. Fourier analysis on the slice
3. The proof
4. Summary

Constant-weight vectors fool low-degree polynomials*

In more detail...

Suppose P is a suitable low-degree polynomial.

If $(X_1, \ldots, X_n) \sim \{0,1\}^n$ and $(Y_1, \ldots, Y_n) \sim \binom{[n]}{n/2}$ then

$P(X_1, \ldots, X_n) \approx P(Y_1, \ldots, Y_n)$.

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$$x_1(1 - n/2) + (x_2 + \cdots + x_n)x_1$$

Suitable polynomials

Definition

A polynomial P is *harmonic* if

$$\sum_{i=1}^{n} \frac{\partial P}{\partial x_i} = 0.$$

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Theorem (Dunkl)

Every function on $\binom{[n]}{n/2}$ *has unique representation as harmonic multilinear polynomial of degree* $\leq n/2$.

Formal statement

For harmonic multilinear polynomial P and test function φ s.t.

- $\deg P = o(\sqrt{n})$
- $\|P\| = 1$
- φ is Lipschitz

holds:

$$\left| \mathop{\mathbb{E}}_{\vec{X} \sim \{0,1\}^n} [\varphi(P(\vec{X}))] - \mathop{\mathbb{E}}_{\vec{Y} \sim \binom{[n]}{n/2}} [\varphi(P(\vec{Y}))] \right| = o(1).$$

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Works also for other slices.

Consequences

Results for $\{0,1\}^n$ proved using classical invariance principle generalize (almost) immediately to the slice.

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- Kindler–Safra theorem for the slice.

Coarse spectral decomposition

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Theorem (Dunkl)

Homogeneous parts are orthogonal on all slices, and so on $\{0,1\}^n$.

L^2 invariance

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Prerequisite for full invariance principle.

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Basis isn't needed to prove the invariance principle.

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Classical invariance principle (MOO)

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Our case

Strategy fails since coordinates of $\binom{[n]}{n/2}$ are not independent!

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- and so similar to its distribution on $\binom{[n]}{n/2-t \leq \cdot \leq n/2+t}$.
- Hence dist. of P on $\binom{[n]}{n/2}$ is similar to its dist. on $\{0,1\}^n$.

Recap

Constant-weight vectors fool
low-degree harmonic multilinear polys

Open questions

Fourier analysis on the slice

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Applications

- Suggestions welcome!

Thanks!

Fourier basis for the slice

Setup

A slice $\binom{[n]}{k}$.

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Basis elements

For every $\ell \leq \min(k, n-k)$ and $b_1 < \cdots < b_\ell \leq n$:

$$\chi_{b_1,\ldots,b_\ell} = \sum_{a_1,\ldots,a_\ell}{}' (x_{a_1} - x_{b_1}) \cdots (x_{a_\ell} - x_{b_\ell}),$$

where:

- $a_1, b_1, \ldots, a_\ell, b_\ell$ all distinct.
- $a_1 < b_1, \ldots, a_\ell < b_\ell$.

Only consider $b_1, \ldots, b_\ell$ for which sum is non-zero.

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Properties

- $\binom{n}{\ell} - \binom{n}{\ell-1}$ basis elements at level ℓ.
- Basis is orthogonal.
- Explicit formula for norm of $\chi_{b_1,\ldots,b_\ell}$.