Toward the KRW conjecture

Cubic Formula Lower Bounds via Communication Complexity

Irit Dinur Or Meir

Formulas

- (de-Morgan) Formulas are circuits with fan-out 1.
- Cannot store intermediate results.
- The formula complexity $\mathsf{L}(f)$ is the size of the smallest formula for f.

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- We now have an optimal analysis of shrinkage.
- So shrinkage is unlikely to get us any further.

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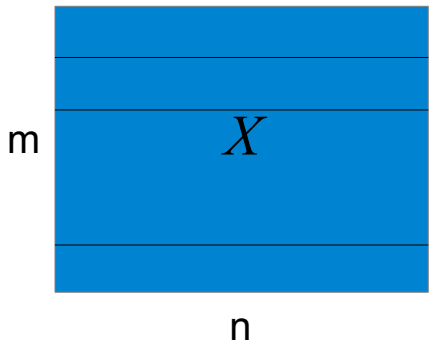

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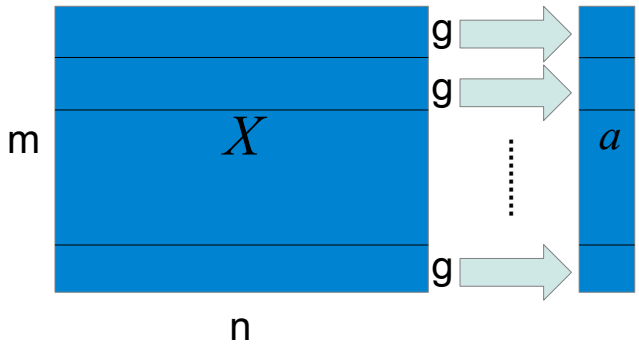

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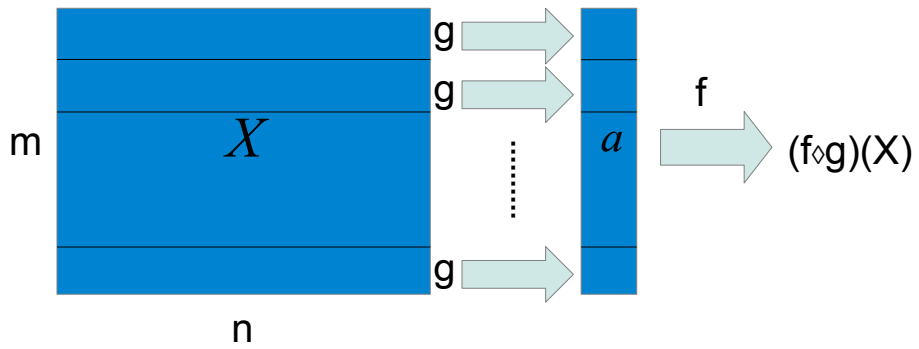

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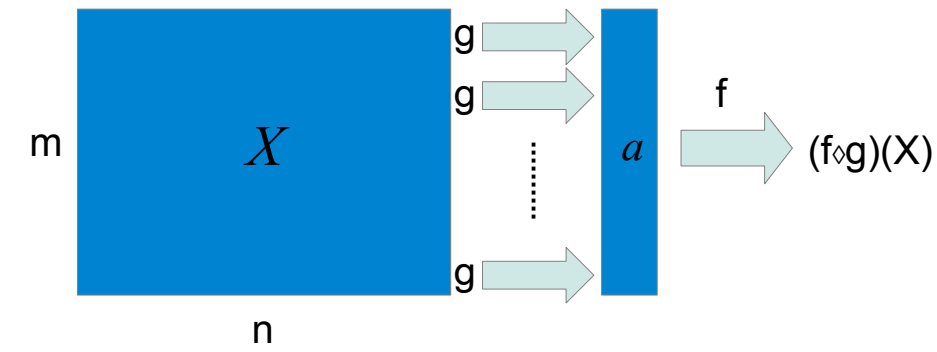

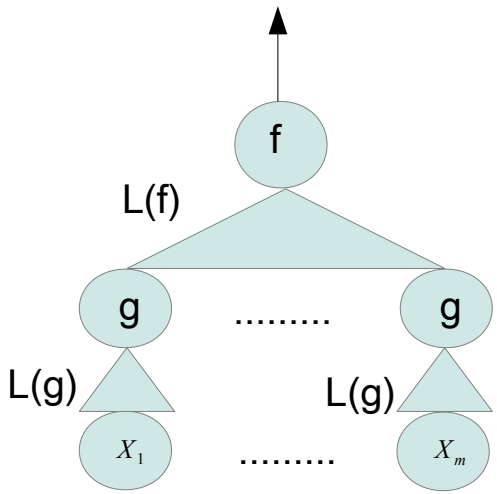

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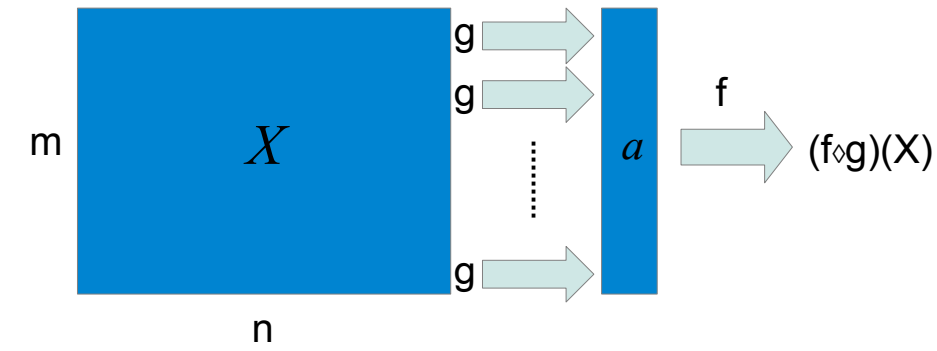

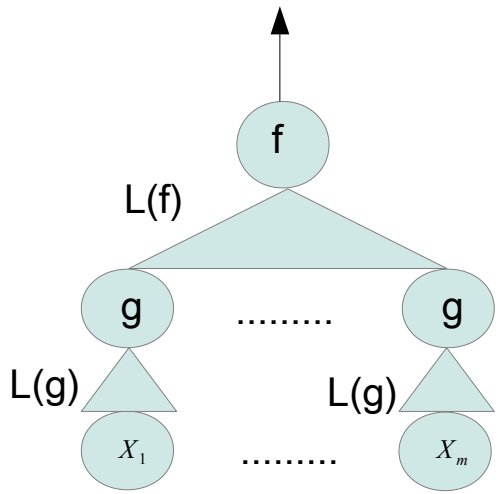

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- Conjecture was proved for $U \diamond U$ by [EIRS91]
- Alternative proof by [HW93].
- Recently: [GMWW14] proved conjecture for $f \diamond U$.

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- We prove conjecture for $f \diamond g$ where g is parity:

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- Actually, the lower bound for $f \diamond g$ already follows from [H98].
- However, our proof is very different, and seems to be more generalizable for other g's.
- Also: new proof of the state-of-the-art cubic lower bound of [H98].

Outline

Karchmer-Wigderson Relations [KW90]

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- This talk: Assume $\mathsf{C}(KW_f) = \log \mathsf{L}(f)$.
- KRW conjecture: $\mathsf{C}(KW_{f \diamond g}) \approx \mathsf{C}(KW_f) + \mathsf{C}(KW_g)$.

KRW and KW

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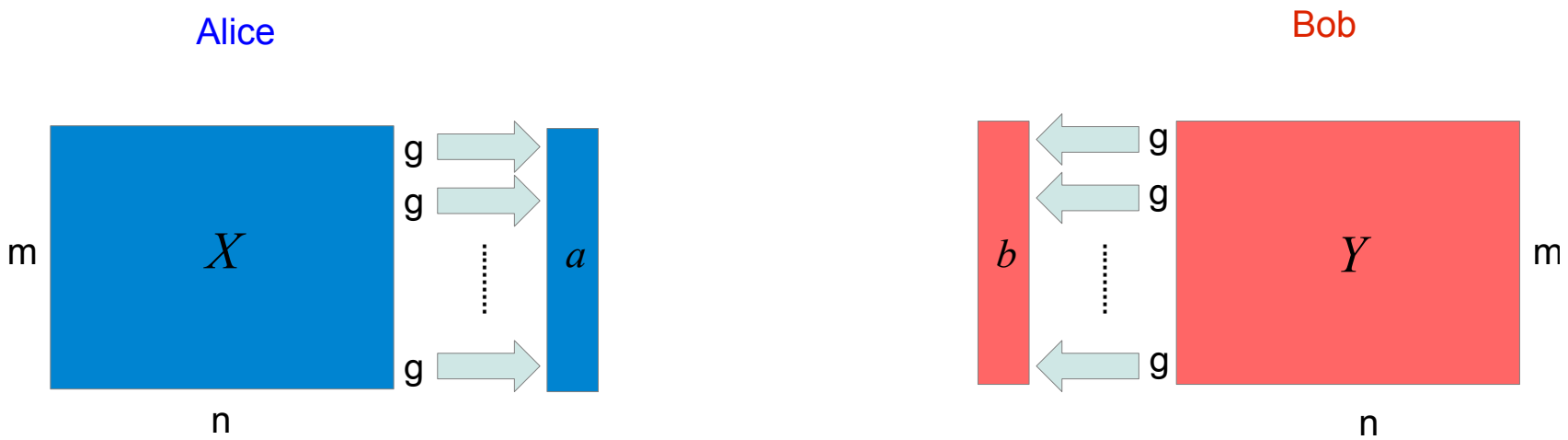

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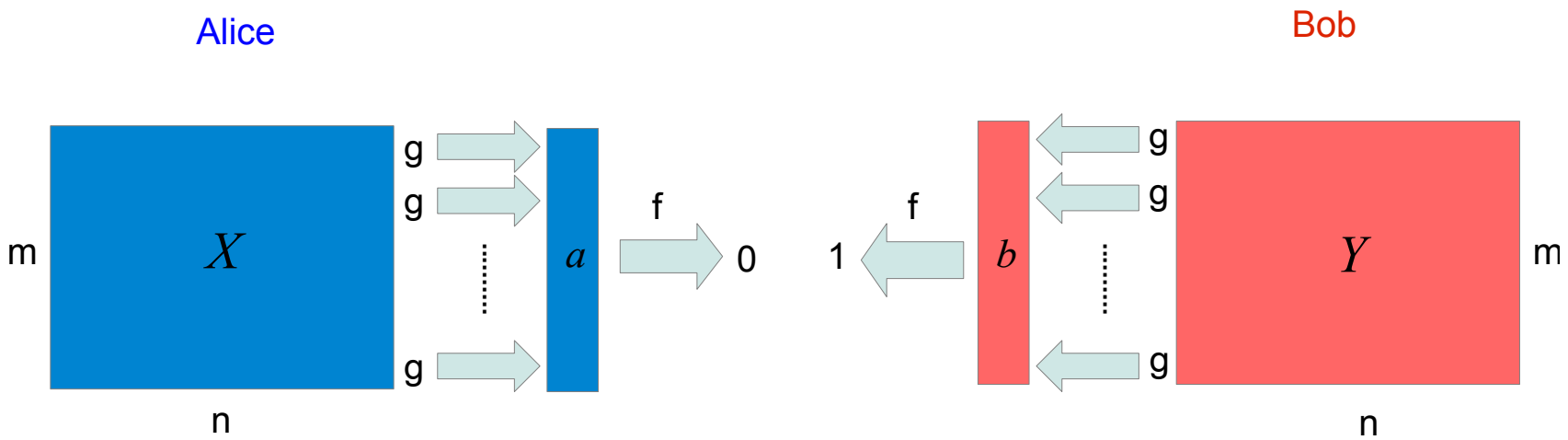

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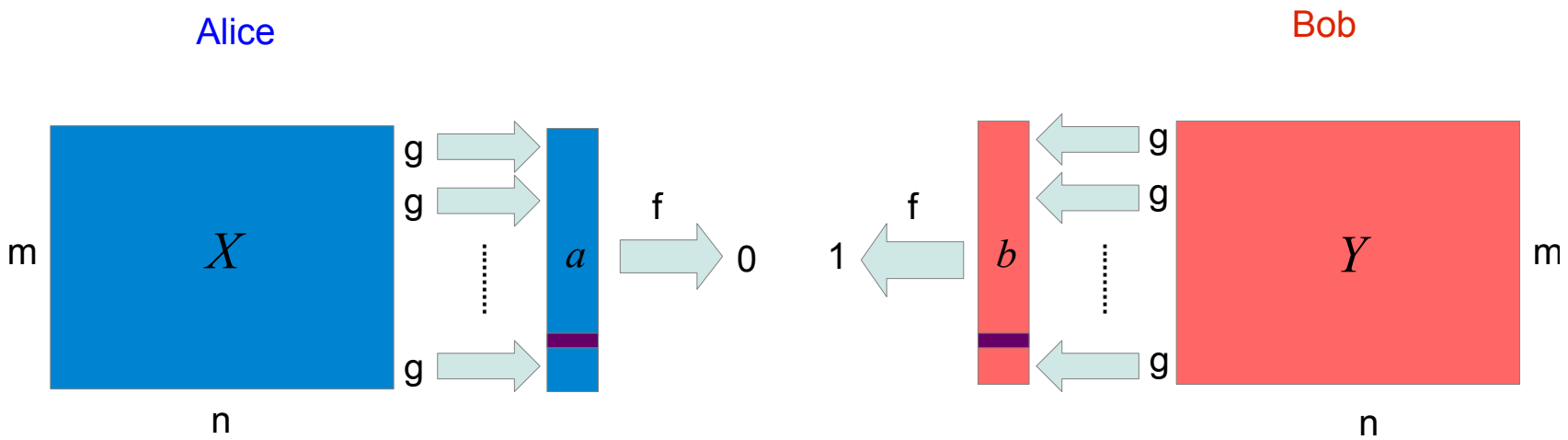

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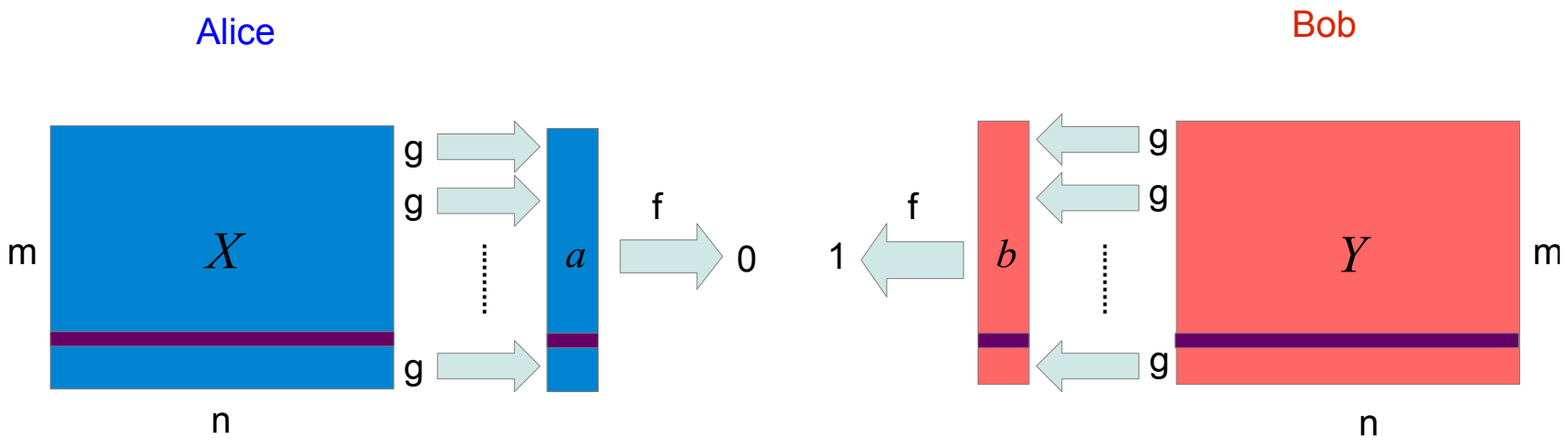

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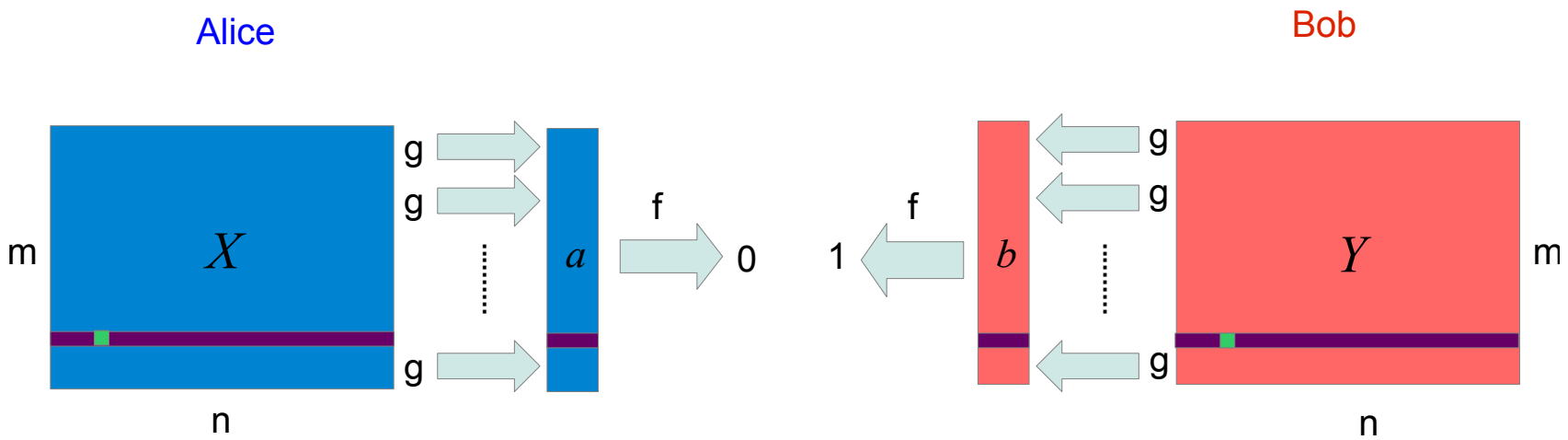

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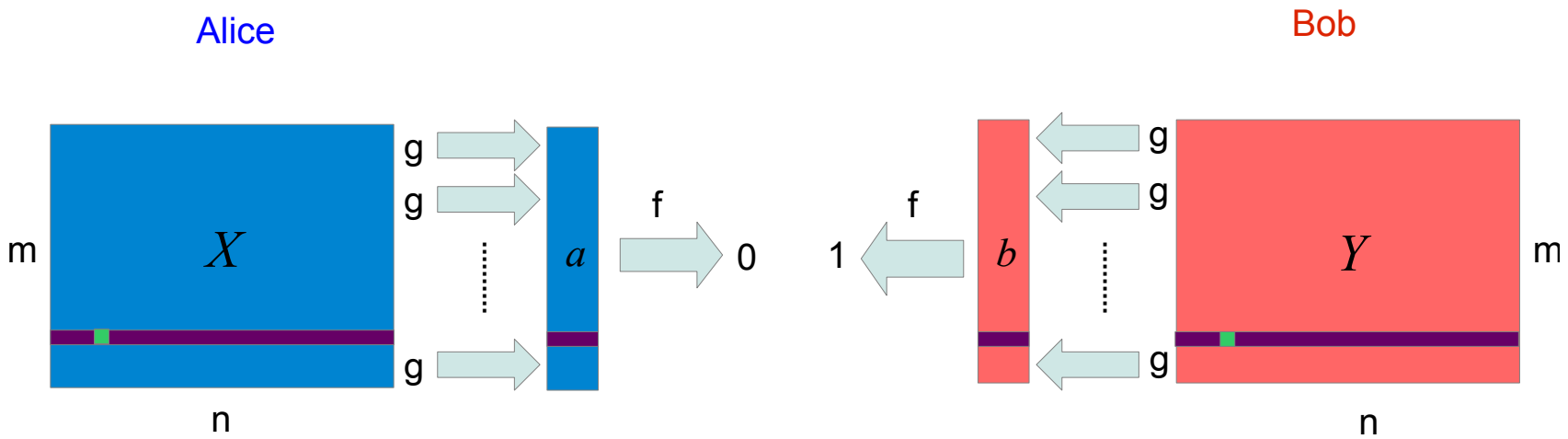

KRW conjecture: $\mathsf{C}(KW_{f \diamond g}) \approx \mathsf{C}(KW_f) + \mathsf{C}(KW_g)$

The obvious protocol is essentially optimal.

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- Fix a protocol Π for $KW_{f \diamond g}$.
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- Partition each transcript π of Π to two parts: $\pi = \pi_1 \circ \pi_2$.
- π_1 and π_2 correspond to the two stages of the obvious protocol.

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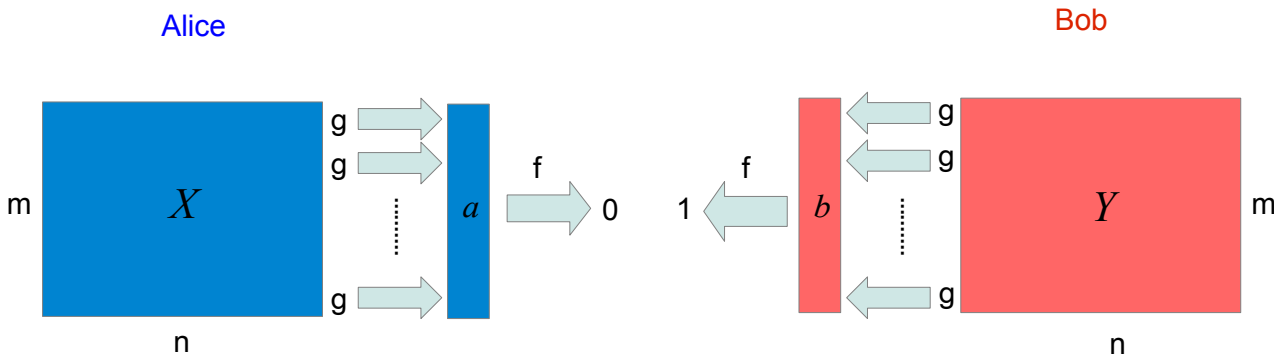
- The first item gives our structural result.
- This will be the focus of the rest of this talk.

Structural Result - Basic Idea



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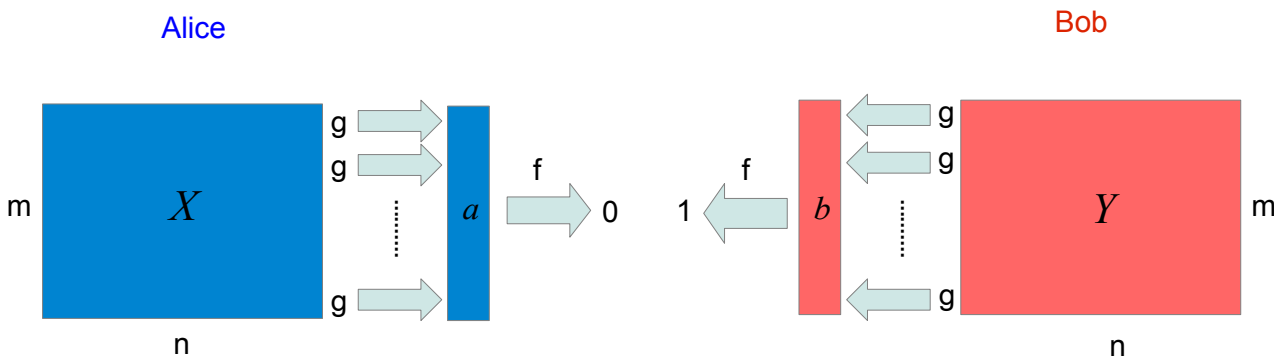

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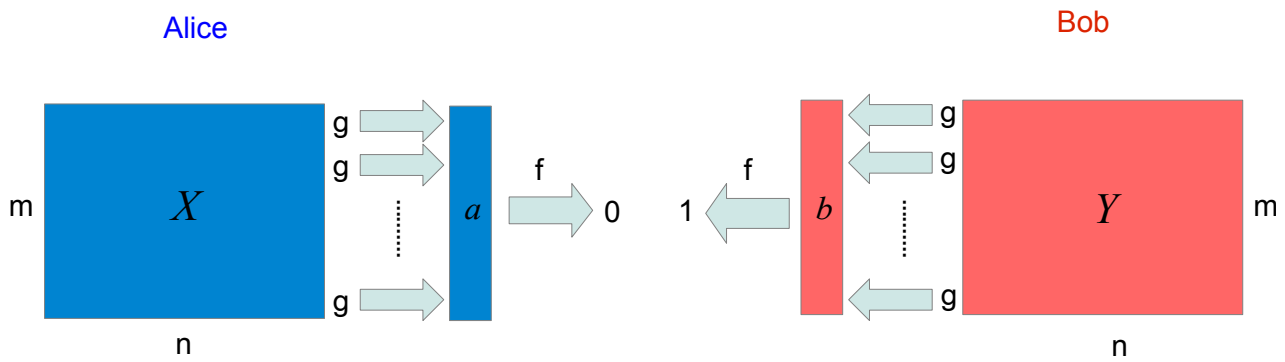

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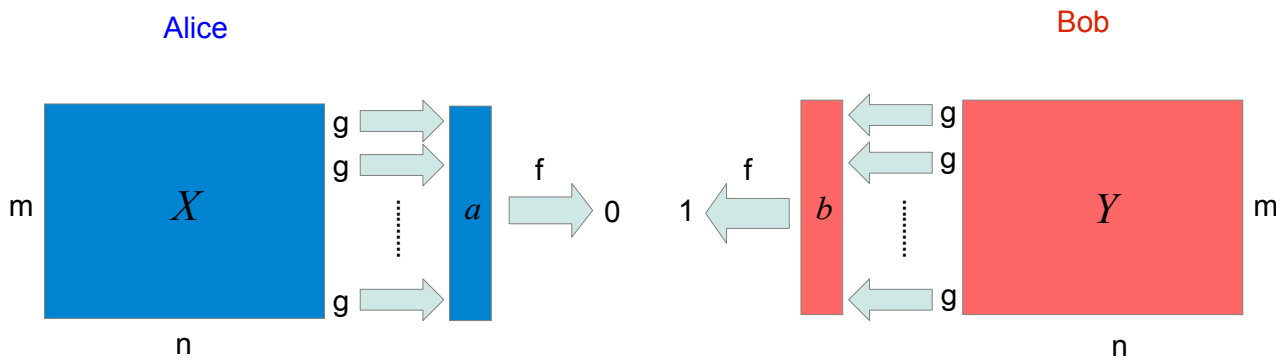

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- However, they already communicated almost $\mathrm{C}(KW_f)$ bits in the 1st stage.
- Need to show: does not make the 2nd stage easier.

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- If they solve a typical row in the 2nd stage, they still need to send $\approx \mathsf{C}(KW_g)$ bits.

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- In particular, need to force $g(X_i) = g(Y_i)$.
- Issue: What if the players already found that $g(X_i) \neq g(Y_i)$ in the first stage?
- This cannot happen, since we know they did not solve KW_f in the first stage.

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- Our novelty: extending this strategy to $f \diamond g$, where $g = \oplus$.
- This setting is considerably more challenging.

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- Direct correspondence between information and complexity:
  - If Alice sends a message that has t bits of information,
  - the complexity of the problem decreases by exactly t bits.
- KW relations are "computational objects".
- There are examples KW_f in which
  - Alice sends little information,
  - and decreases the complexity by a lot.

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- For KW_f, we prove a general "fortification lemma", which creates a correspondence between information and complexity.
- For KW_g, we use the fact that the lower bound for parity has an information-theoretic proof.
- Therefore, can analyze directly how information and complexity interact for parity.

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- Also, most rectangle bounds don't work [KKN95].

- Our work is the first to bypass this barrier.

Outline

Fortification

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- Can fix it: discard in advance all Alice's inputs that start with 1.

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Fortification lemma

Every rectangle $A \times B$ has a fortified subrectangle whose complexity is close to that of $A \times B$.

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- This work: we prove it for min-entropy.

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- Result: We (re-)prove the KRW conjecture for the special case $f \diamond g$ where g is parity.
- Corollary: an alternative proof for the cubic lower bound for formulas.
- New tools: fortification lemma, averaging argument for min-entropy.
- Future direction: replace $\bigoplus$ with a general function?

The 1-out-of-m problem [BBKW14]

- Alice and Bob are given m distinct instances of KW_g.
- They need to solve one of them.
- Prove: not easier than solving a single instance.

Thank you!