New hardness results for graph and hypergraph colorings

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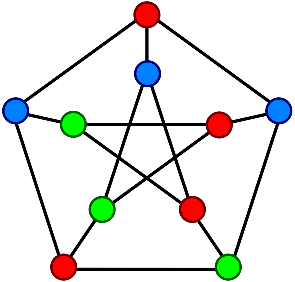
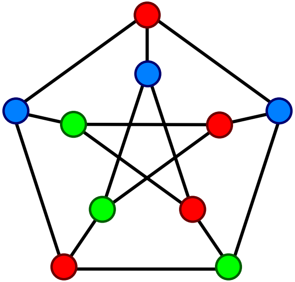
CCC 2016



Source: Wikipedia

Graph Coloring



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Determining if a graph can be colored with t colors is NP*-complete when* $t \geq 3$.

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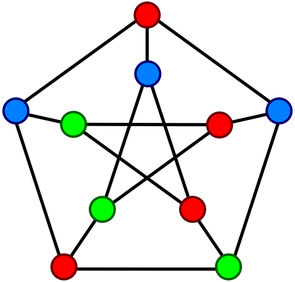

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Theorem (Karp, 1972)

Determining if a graph can be colored with t colors is NP*-complete when* $t \geq 3$.

If given that the graph is t-colorable, NP-hard to find t-coloring.

Approximate Graph Coloring

Question (Approximate Coloring–search version)

Can a t-colorable graph with n vertices be efficiently colored with $c(n)$-colors, where $c(n) \geq t \geq 3$?

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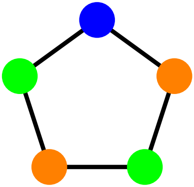

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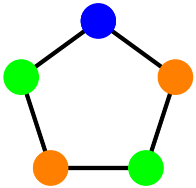

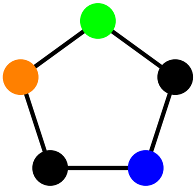

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Question (Approximate Coloring–decision version)

Given a graph G on n vertices,

- *Output* **YES** *if G can be colored with t colors,*
- *Output* **NO** *if G cannot be colored with $c(n)$ colors.*

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Result generalizes to $c = 2t - 2$.

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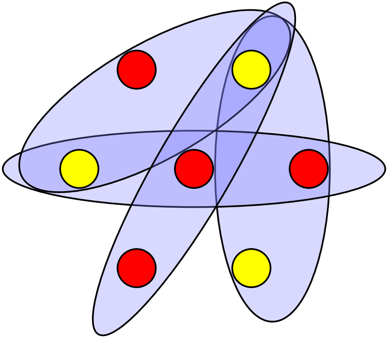

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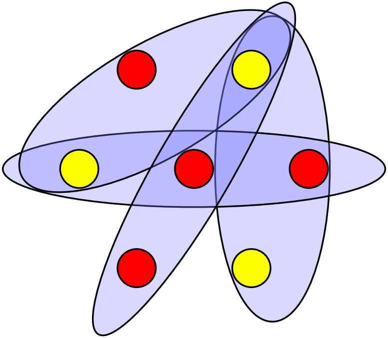

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Hypergraph is c-colorable if each hyperedge is not monochromatic.

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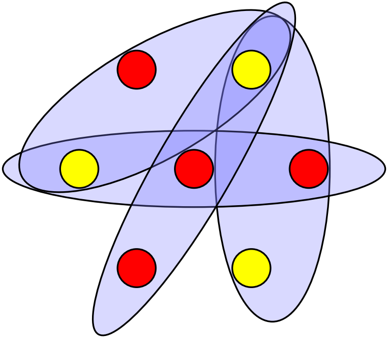

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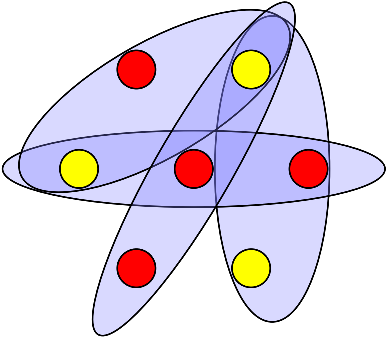

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Given a k-uniform 2-colorable hypergraph, can we efficiently color it with c colors so no hyperedge is monochromatic?

Hypergraph coloring

- Algorithms ($k = 3$)
  - $c = \tilde{O}(n^{2/9})$ (Alon, Kelsen, Mahajan, Ramesh, 1996; Chen, Frieze, 1996)
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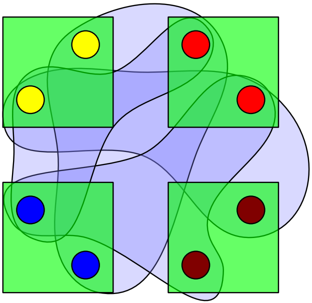
- Since strong hardness results are known, we consider a problem with an even stronger promise.

Strong hypergraph coloring



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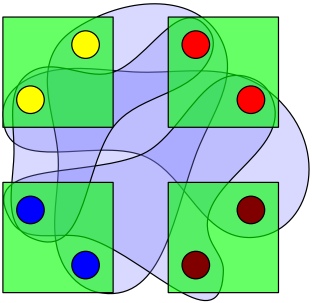

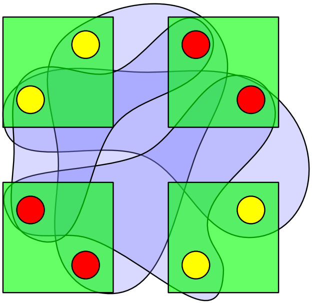

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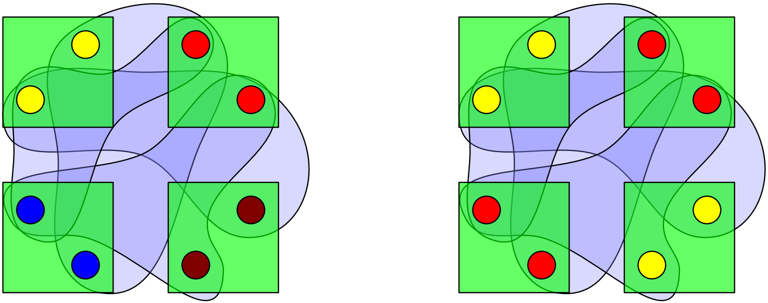

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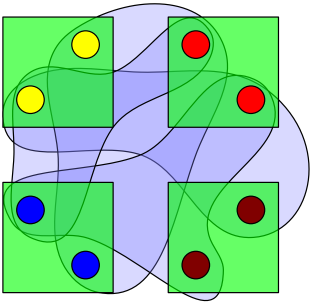

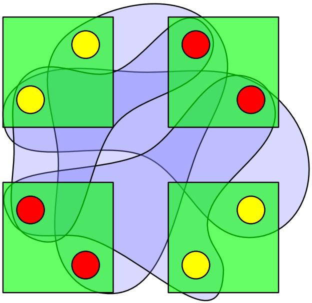

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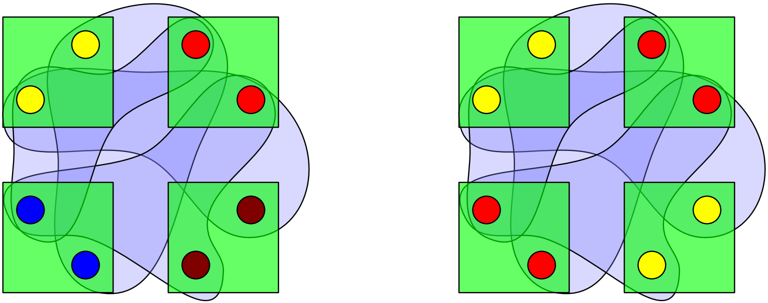

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Not told t-partite structure beforehand.

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Theorem (Brakensiek, Guruswami)

NP*-hard to color a k-uniform, t-partite hypergraph with 2 colors when $t = \lceil \frac{3k}{2} \rceil$.*

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   - Often used in hardness of approximation reductions.

Gadget Motivation: CSP theory

Consists of *variables* V and *constraints* $R_1, \ldots, R_k$ over *domain* D.

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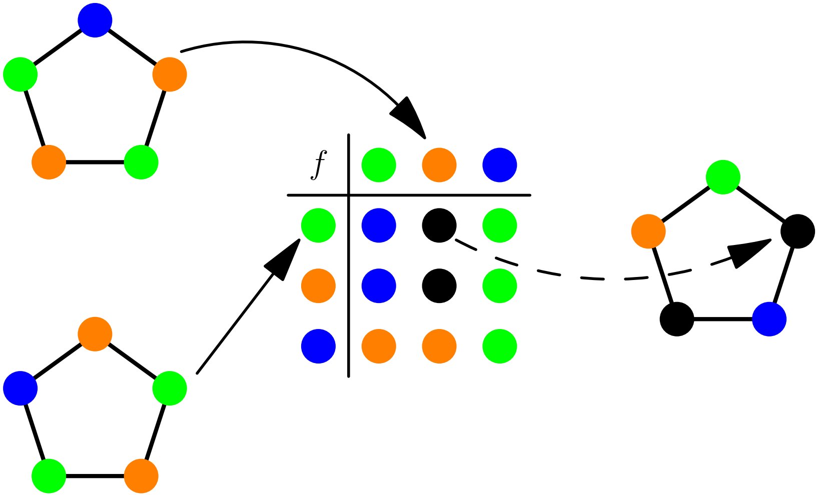
For promise problems, need *weak* polymorphisms (Austrin, Guruswami, Håstad; 2014)

Gadget: Weak Polymorphisms



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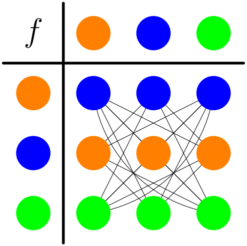
Recall: Show it is NP-hard to c-color t-colorable graphs.



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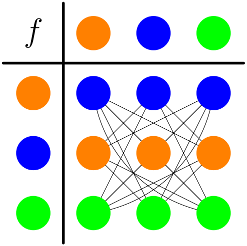

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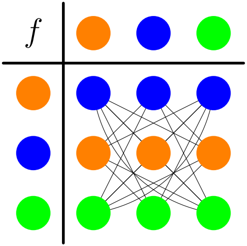

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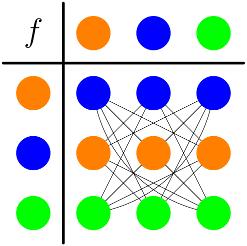

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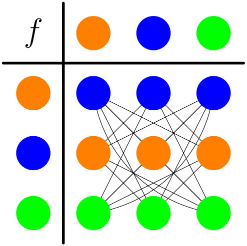

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- Coloring: $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$
- Infer the label by *decoding* f.

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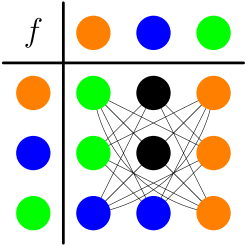

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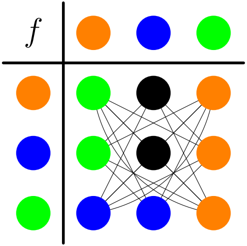

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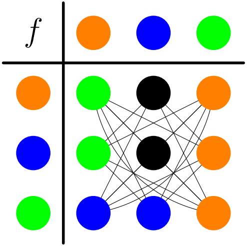

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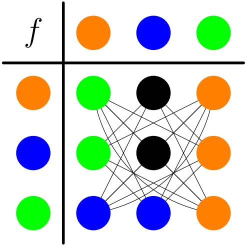

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- Expected by algebraic theory for CSPs.

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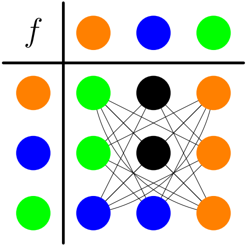

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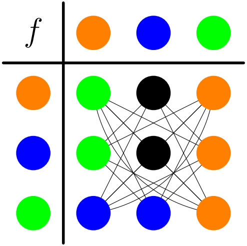

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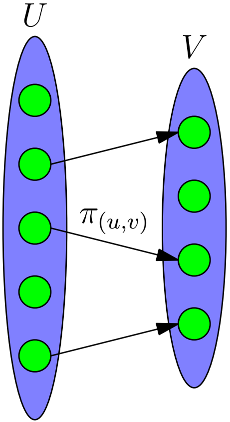

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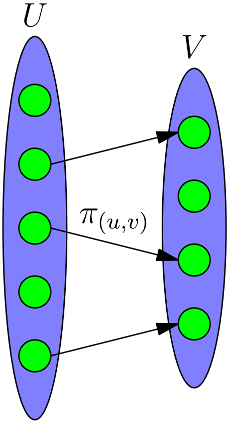

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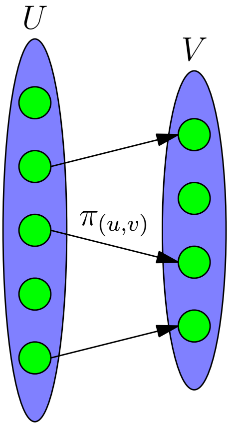

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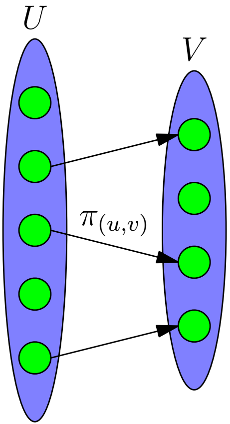

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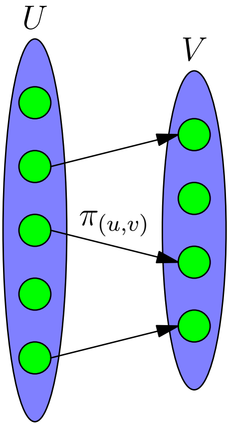

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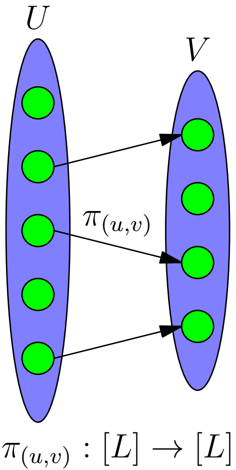

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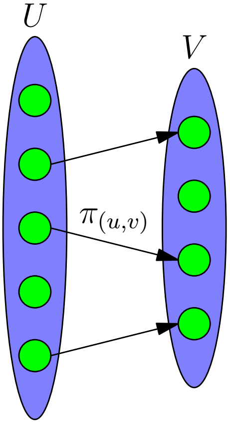
Theorem (Label Cover)

For all $\epsilon > 0$, exists L where distinguishing
YES: *exists σ_1, σ_2 satisfying all edges.*
NO: *all σ_1, σ_2 satisfy at most ϵ of the edges.*
is NP-hard.

Outer Verifier + Inner Verifier



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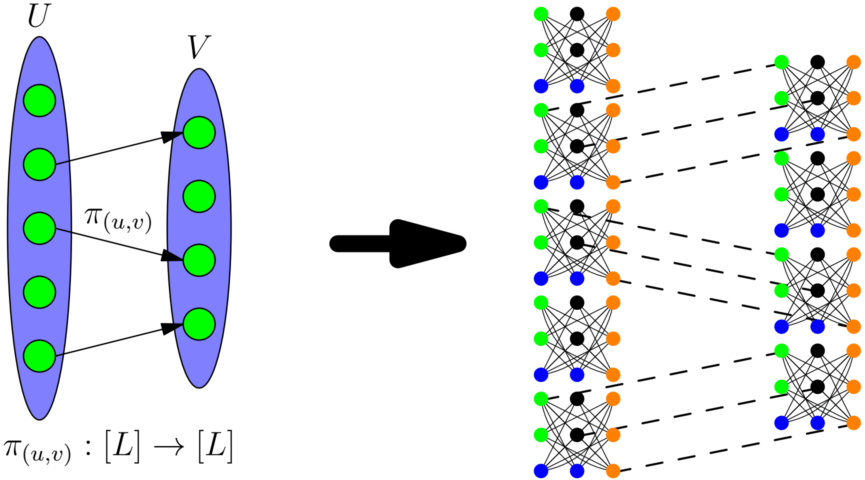

Graph Coloring: Reduction

Lemma (Near-dictatorship)

If $c \leq 2t - 2$*, there exists unique* $i \in \{1, \ldots, L\}$ *and* $S \subseteq \mathbb{Z}_c$ *of size* t *such that if* $f(x) = f(y) \in S$ *then* $x_i = y_i$

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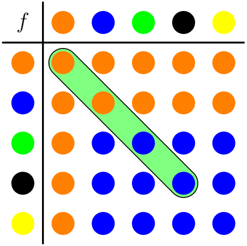
Thus, c-coloring a t-colorable graph is NP-hard if $c \leq 2t-2$.

Hardness of c-coloring a k-uniform, t-partite hypergraph.



$t = 5, k = 4, c = 2$

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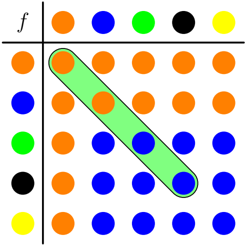

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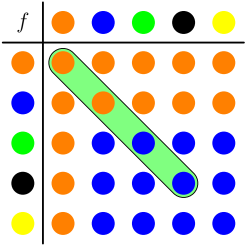

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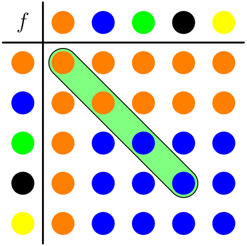

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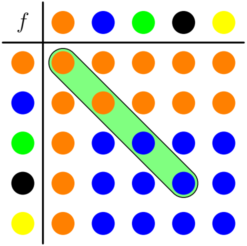

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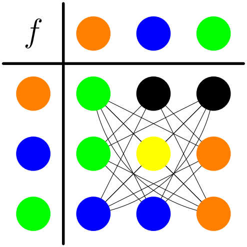
Lemma (Brakensiek, Guruswami)

If $c = 2$ and $t = \lceil \frac{3k}{2} \rceil$, there are injective maps $g_i : \mathbb{Z}_{2\lceil \frac{k}{2} \rceil + 2} \to \mathbb{Z}_t$ $(i \in L)$ and a dictator $g : \mathbb{Z}^L_{2\lceil \frac{k}{2} \rceil + 2} \to \mathbb{Z}_t$ such that $g(x_1, \ldots, x_L) = f(g_1(x_1), g_2(x_2), \ldots, g_L(x_L))$.

Limitations



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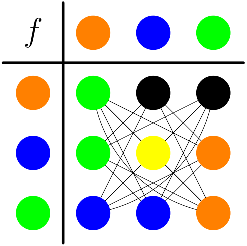

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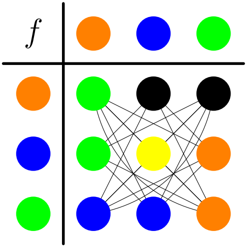

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Open Questions

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- Case $c = 2$ may be in reach with current techniques.

Thank you!